

KÄHLER DIFFERENTIALS FOR FAT POINT SCHEMES IN $\mathbb{P}^1 \times \mathbb{P}^1$

ELENA GUARDO, MARTIN KREUZER, TRAN N. K. LINH, AND LE NGOC LONG

ABSTRACT. Let \mathbb{X} be a set of K -rational points in $\mathbb{P}^1 \times \mathbb{P}^1$ over a field K of characteristic zero, let \mathbb{Y} be a fat point scheme supported at \mathbb{X} , and let $R_{\mathbb{Y}}$ be the bihomogeneous coordinate ring of \mathbb{Y} . In this paper we investigate the module of Kähler differentials $\Omega_{R_{\mathbb{Y}}/K}^1$. We describe this bigraded $R_{\mathbb{Y}}$ -module explicitly via a homogeneous short exact sequence and compute its Hilbert function in a number of special cases, in particular when the support \mathbb{X} is a complete intersection or an almost complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, we introduce a Kähler different for \mathbb{Y} and use it to characterize reduced fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ having the Cayley-Bacharach property.

1. INTRODUCTION

In [3], G. de Dominicis and the second author introduced some methods using algebraic differential forms into the study of 0-dimensional subschemes of \mathbb{P}^n . More precisely, given a 0-dimensional subscheme \mathbb{X} of the projective n -space \mathbb{P}^n over a field K of characteristic zero with homogeneous vanishing ideal $I_{\mathbb{X}}$ in $R = K[X_0, \dots, X_n]$ and homogeneous coordinate ring $R_{\mathbb{X}} = R/I_{\mathbb{X}}$, let J be the kernel of the multiplication map $\mu : R_{\mathbb{X}} \otimes_K R_{\mathbb{X}} \rightarrow R_{\mathbb{X}}$. Then the module of Kähler differentials of $R_{\mathbb{X}}/K$ is the $R_{\mathbb{X}}$ -module $\Omega_{R_{\mathbb{X}}/K}^1 = J/J^2$. The structure of this module can be described using the canonical exact sequence

$$0 \rightarrow I_{\mathbb{X}}^{(2)}/I_{\mathbb{X}}^2 \rightarrow I_{\mathbb{X}}/I_{\mathbb{X}}^2 \rightarrow R_{\mathbb{X}}^{n+1}(-1) \rightarrow \Omega_{R_{\mathbb{X}}/K}^1 \rightarrow 0$$

which follows from [20, Prop. 4.13]. For instance, if \mathbb{X} is the complete intersection of hypersurfaces of degrees d_1, \dots, d_n then it follows that the Hilbert function of $\Omega_{R_{\mathbb{X}}/K}^1$ is given by $\text{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(i) = (n+1)\text{HF}_{\mathbb{X}}(i-1) - \sum_{j=1}^n \text{HF}_{\mathbb{X}}(i-d_j)$ for all $i \in \mathbb{Z}$ (see [3, Prop. 4.3]). Later, in [16], these differential algebra techniques were extended to fat point schemes in \mathbb{P}^n .

In this paper we examine the natural question of whether these differential algebraic methods can be applied to study 0-dimensional subschemes \mathbb{Y} of $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, if \mathbb{Y} is a fat point scheme, if $S = K[X_0, X_1, Y_0, Y_1]$, and if $R_{\mathbb{Y}} = S/I_{\mathbb{Y}}$ is the bihomogeneous coordinate ring of \mathbb{Y} , we show that the module of Kähler differentials $\Omega_{R_{\mathbb{Y}}/K}^1$ contains a significant amount of information about \mathbb{Y} .

The paper is structured as follows. In Section 2 we fix the notation and recall a number of results about 0-dimensional subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$ which we use later on (cf. [8, 11, 12, 14]). In particular, we recall the definition of an ACM subscheme,

Date: November 29, 2016.

1991 Mathematics Subject Classification. Primary 13N05, Secondary 13D40, 14N05, 13C40.

Key words and phrases. fat point scheme, ACM fat point scheme, separators, Kähler differentials, Kähler different, Hilbert function, complete intersection.

separators, minimal separators, and the degree tuple of a set of minimal separators for a fat point subscheme in $\mathbb{P}^1 \times \mathbb{P}^1$.

In Section 3 we introduce the main object of the study of this paper, namely the Kähler differential module $\Omega_{R_{\mathbb{Y}}/K}^1$ for the bihomogeneous coordinate ring $R_{\mathbb{Y}}$ of a fat point scheme \mathbb{Y} in $\mathbb{P}^1 \times \mathbb{P}^1$. Based on the general theory in [20], one can describe this module via generators and relations. Our first main result is Theorem 3.5 which contains a more explicit presentation of the module $\Omega_{R_{\mathbb{Y}}/K}^1$ via an exact sequence

$$0 \longrightarrow I_{\mathbb{Y}}/I_{\mathbb{V}} \longrightarrow R_{\mathbb{Y}}^2(-1, 0) \oplus R_{\mathbb{Y}}^2(0, -1) \longrightarrow \Omega_{R_{\mathbb{Y}}/K}^1 \longrightarrow 0$$

where \mathbb{V} is the fat point scheme obtained by increasing the multiplicities of all points in \mathbb{Y} by one. This exact sequence shows that one can compute the Hilbert function of the Kähler differential module of $R_{\mathbb{Y}}/K$ from the Hilbert functions of \mathbb{Y} and \mathbb{V} .

Section 4 contains a more detailed study of the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ for a fat point scheme \mathbb{Y} in $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, if \mathbb{X} is a set of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{Y} is the fat point scheme $\mathbb{Y} = \{(P_{ij}, m_{ij}) \mid P_{ij} \in \mathbb{X}\}$, we can associate to \mathbb{Y} two tuples $\alpha_{\mathbb{Y}}$ and $\beta_{\mathbb{Y}}$ as in [8], § 3. Using these, we define two new tuples $\hat{\alpha}_{\mathbb{Y}}$ and $\hat{\beta}_{\mathbb{Y}}$ and give formulas for the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ depending only on $\hat{\alpha}_{\mathbb{Y}}$ and $\hat{\beta}_{\mathbb{Y}}$ (see Prop. 4.3). These formulas allow us to describe that Hilbert function in large degrees explicitly and to show that it is determined by finitely many of its values. If the corresponding fat point scheme $\mathbb{V} = \{(P_{ij}, m_{ij} + 1) \mid P_{ij} \in \mathbb{X}\}$ is arithmetically Cohen-Macaulay (and hence, if $m_{ij} > 1$ for some i, j , also \mathbb{Y} is arithmetically Cohen-Macaulay), we can describe the Hilbert function of the Kähler differential module of \mathbb{Y} in terms of $\alpha_{\mathbb{Y}}$ and $\hat{\alpha}_{\mathbb{Y}}$ (see Prop. 5.4).

For a fat point scheme \mathbb{Y} whose support \mathbb{X} is a complete or an almost complete intersection, we give even more detailed descriptions of the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ in Section 5. In particular, if \mathbb{X} is a reduced complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{Y} is an equimultiple fat point scheme supported at \mathbb{X} , we determine this Hilbert function explicitly (see Prop. 5.6). Furthermore, if $m \geq 2$ and if $\mathbb{Y} = m\mathbb{X}$ is supported at a complete intersection \mathbb{X} , we show that the Hilbert function of the Kähler differential module $\Omega_{R_{\mathbb{Y}}/K}^1$ of $\mathbb{Y}_{ij} = \sum_{(k,l) \neq (i,j)} mP_{kl} + (m-1)P_{ij}$ does not depend on the choice of (i, j) (see Prop. 5.10). In the case $m = 1$, the same result holds if \mathbb{X} is a complete intersection of type (h, h) (see Prop. 5.11). Of course, this independence on (i, j) is reminiscent of the Cayley-Bacharach Property (CBP) in $\mathbb{P}^1 \times \mathbb{P}^1$ which was introduced by the first author in [6] and is studied here in the last section.

But before, in Section 6, we look at the Kähler different of a fat point scheme \mathbb{Y} in $\mathbb{P}^1 \times \mathbb{P}^1$. For this we restrict our attention to arithmetically Cohen-Macaulay (ACM for short) fat point schemes. Thus we may assume that x_0, y_0 give rise to a regular sequence in $R_{\mathbb{Y}}$. Then the initial Fitting ideal $\vartheta_{\mathbb{Y}} = F_0(\Omega_{R_{\mathbb{Y}}/K[x_0, y_0]}^1)$ is also called the *Kähler different* of \mathbb{Y} w.r.t. $\{x_0, y_0\}$. In Prop. 6.2 we provide some basic properties of the Hilbert function of $\vartheta_{\mathbb{Y}}$ and show that $\text{HF}_{\vartheta_{\mathbb{Y}}} = 0$ if and only if \mathbb{Y} contains no reduced point. For an ACM set of reduced points, even more properties of the Hilbert function of $\vartheta_{\mathbb{Y}}$ are shown in Prop. 6.3.

Finally, in the last section, we define the Cayley-Bacharach property for a set \mathbb{X} of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$. If \mathbb{X} is ACM, we characterize this property using the Kähler different, and we write down the Hilbert function of $\vartheta_{\mathbb{X}}$.

Unless explicitly mentioned otherwise, we use the definitions and notation regarding Kähler differential modules, Kähler differentials, arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their properties found in the books [15, 18, 19, 20]. All examples were computed using the computer algebra system ApCoCoA (see [1]).

2. ZERO-DIMENSIONAL SCHEMES IN $\mathbb{P}^1 \times \mathbb{P}^1$

Let K be a field of characteristic zero, and let $S = K[X_0, X_1, Y_0, Y_1]$ be the bigraded polynomial ring equipped with the \mathbb{Z}^2 -grading defined by $\deg X_0 = \deg X_1 = (1, 0)$ and $\deg Y_0 = \deg Y_1 = (0, 1)$. For $(i, j) \in \mathbb{Z}^2$, we let $S_{i,j}$ be the homogeneous component of degree (i, j) of S , i.e., the K -vector space with basis $\{X_0^{\alpha_0} X_1^{\alpha_1} Y_0^{\beta_0} Y_1^{\beta_1} \mid \alpha_0 + \alpha_1 = i, \beta_0 + \beta_1 = j, \alpha_k, \beta_\ell \in \mathbb{N}\}$.

Note that $0 \in S_{i,j}$ for all i, j , and all elements of K have degree $(0, 0)$. Given two tuples $(i_1, j_1), (i_2, j_2) \in \mathbb{Z}^2$ we write $(i_1, j_1) \preceq (i_2, j_2)$ if $i_1 \leq i_2$ and $j_1 \leq j_2$.

Given an ideal $I \subseteq S$, we set $I_{i,j} := I \cap S_{i,j}$ for all $(i, j) \in \mathbb{Z}^2$. Clearly, $I_{i,j}$ is a K -vector subspace of $S_{i,j}$ and $I \supseteq \bigoplus_{(i,j) \in \mathbb{Z}^2} I_{i,j}$. The ideal I is called *bihomogeneous* if $I = \bigoplus_{(i,j) \in \mathbb{Z}^2} I_{i,j}$. If I is a bihomogeneous ideal of S then the quotient ring S/I also inherits the structure of a bigraded ring via $(S/I)_{i,j} := S_{i,j}/I_{i,j}$ for all $(i, j) \in \mathbb{Z}^2$.

A finitely generated S -module M is a *bigraded S -module* if it has a direct sum decomposition

$$M = \bigoplus_{(i,j) \in \mathbb{Z}^2} M_{i,j}$$

with the property that $S_{i,j} M_{k,l} \subseteq M_{i+k, j+l}$ for all $(i, j), (k, l) \in \mathbb{Z}^2$. For a bihomogeneous ideal I of S , both I and S/I are examples of bigraded S -modules. Another example is the polynomial ring S equipped with a *shifted grading*. Specifically, for $(a, b) \in \mathbb{Z}^2$, we let

$$S(-a, -b)_{i,j} := S_{i-a, j-b}.$$

Notice that $S_{i,j} = 0$ if $(i, j) \not\geq (0, 0)$.

Definition 2.1. Let M be a finitely generated bigraded S -module. The *Hilbert function* of M is the numerical function $\text{HF}_M : \mathbb{Z}^2 \rightarrow \mathbb{N}$ defined by

$$\text{HF}_M(i, j) := \dim_K M_{i,j} \quad \text{for all } (i, j) \in \mathbb{Z}^2.$$

In particular, for a bihomogeneous ideal I of S , the Hilbert function of S/I satisfies

$$\text{HF}_{S/I}(i, j) := \dim_K (S/I)_{i,j} = \dim_K S_{i,j} - \dim_K I_{i,j} \quad \text{for all } (i, j) \in \mathbb{Z}^2.$$

If M is a finitely generated bigraded S -module such that $\text{HF}_M(i, j) = 0$ for $(i, j) \not\geq (0, 0)$, we write the Hilbert function of M as an infinite matrix, where the initial row and column are indexed by 0.

Example 2.2. In the polynomial ring S , the set of terms $\{X_0^a X_1^b Y_0^c Y_1^d \mid a + b = i, c + d = j\}$ forms a basis of the K -vector space $S_{i,j}$ for all $(i, j) \geq (0, 0)$. So, the Hilbert function of S satisfies $\text{HF}_S(i, j) = 0$ for $(i, j) \not\geq (0, 0)$ and

$$\text{HF}_S(i, j) = \dim_K S_{i,j} = (i+1)(j+1)$$

for all $(i, j) \succeq (0, 0)$. In this case, we can write

$$\mathrm{HF}_S = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots \\ 2 & 4 & 6 & 8 & \cdots \\ 3 & 6 & 9 & 12 & \cdots \\ 4 & 8 & 12 & 16 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Next let us consider the monomial ideal $I = \langle X_1, Y_1 \rangle$ in S . We have $\dim_K I_{0,0} = 0$, and $\dim_K I_{i,j} = (i+1)(j+1) - 1$ for $(i, j) \in \mathbb{N}^2 \setminus \{(0, 0)\}$ since every term of degree (i, j) is an element of $I_{i,j}$, except for $X_0^i Y_0^j$. So, the Hilbert function of S/I is given by

$$\mathrm{HF}_{S/I} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Recall that a point in $\mathbb{P}^1 \times \mathbb{P}^1$ is of the form

$$P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$$

where $[a_0 : a_1], [b_0 : b_1] \in \mathbb{P}^1$. Its vanishing ideal is the bihomogeneous prime ideal

$$I_P = \langle a_1 X_0 - a_0 X_1, b_1 Y_0 - b_0 Y_1 \rangle.$$

Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projection morphism given by $P_1 \times P_2 \mapsto P_1$ and let $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projection morphism given by $P_1 \times P_2 \mapsto P_2$.

Let \mathbb{X} be a set of s distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$. The set $\pi_1(\mathbb{X}) = \{Q_1, \dots, Q_r\}$ is the set of $r \leq s$ distinct first components of the points of \mathbb{X} . Similarly, the set $\pi_2(\mathbb{X}) = \{R_1, \dots, R_t\}$ is the set of $t \leq s$ distinct second components. For $i = 1, \dots, r$, let L_{Q_i} denote the $(1, 0)$ -form (i.e., the linear form in $K[X_0, X_1]$) which vanishes at all points of $\mathbb{P}^1 \times \mathbb{P}^1$ whose first component is Q_i . Similarly, for $j = 1, \dots, t$, let L_{R_j} denote the $(0, 1)$ -form that vanishes at all the points of $\mathbb{P}^1 \times \mathbb{P}^1$ with second component R_j . Furthermore, we let

$$D_{\mathbb{X}} := \{ (i, j) \mid P_{ij} = Q_i \times R_j \in \mathbb{X} \}.$$

Definition 2.3. For $(i, j) \in D_{\mathbb{X}}$, let m_{ij} be a positive integer, and let \wp_{ij} be the vanishing ideal of the point $P_{ij} \in \mathbb{X}$. Let \mathbb{Y} be the subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the saturated bihomogeneous ideal

$$I_{\mathbb{Y}} = \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}}.$$

a) The scheme \mathbb{Y} is called a *fat point scheme* of $\mathbb{P}^1 \times \mathbb{P}^1$ and written as

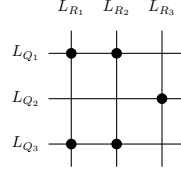
$$\mathbb{Y} = \{ (P_{ij}; m_{ij}) \mid (i, j) \in D_{\mathbb{X}} \} \quad \text{or} \quad \mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}.$$

b) The integer m_{ij} is called the *multiplicity* of the point P_{ij} in \mathbb{Y} .

c) If $m_{ij} = m$ for all $(i, j) \in D_{\mathbb{X}}$, we denote \mathbb{Y} also by $m\mathbb{X}$ and call it an *equimultiple* (or *homogeneous*) fat point scheme.

The bihomogeneous coordinate ring of a fat point scheme \mathbb{Y} in $\mathbb{P}^1 \times \mathbb{P}^1$ is given by $R_{\mathbb{Y}} := S/I_{\mathbb{Y}}$ and its Hilbert function will be denoted by $\text{HF}_{\mathbb{Y}}$. Also, the support of \mathbb{Y} is $\mathbb{X} = \text{Supp}(\mathbb{Y})$.

Example 2.4. Let $i \geq 0$, let $Q_i = R_i = [1 : i] \in \mathbb{P}^1$, let P_{ij} denote the point $Q_i \times R_j$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathbb{X} = \{P_{11}, P_{12}, P_{23}, P_{31}, P_{32}\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. The point P_{ij} can be viewed as the intersection of the horizontal ruling defined by $L_{Q_i} = X_1 - iX_0$ and the vertical ruling defined by $L_{R_j} = Y_1 - jY_0$. By regarding $\mathbb{P}^1 \times \mathbb{P}^1$ as a grid with horizontal and vertical rulings (see [15, Section 3.2]), we can sketch the set \mathbb{X} as in the following figure.



Let \mathbb{Y} and \mathbb{V} be the two fat point schemes supported at \mathbb{X}

$$\mathbb{Y} = 2P_{11} + 2P_{12} + P_{23} + P_{31} + 2P_{32} \text{ and } \mathbb{V} = 3P_{11} + 3P_{12} + 2P_{23} + 2P_{31} + 3P_{32}.$$

Then, using ApCoCoA, the Hilbert functions of \mathbb{Y} and of \mathbb{V} are

$$\text{HF}_{\mathbb{Y}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 5 & \dots \\ 2 & 4 & 6 & 8 & 8 & 8 & \dots \\ 3 & 6 & 9 & 10 & 10 & 10 & \dots \\ 4 & 8 & 10 & 11 & 11 & 11 & \dots \\ 5 & 8 & 10 & 11 & 11 & 11 & \dots \\ 5 & 8 & 10 & 11 & 11 & 11 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\text{HF}_{\mathbb{V}} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8 & \dots \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 14 & 14 & \dots \\ 3 & 6 & 9 & 12 & 15 & 18 & 18 & 18 & 18 & \dots \\ 4 & 8 & 12 & 16 & 20 & 21 & 21 & 21 & 21 & \dots \\ 5 & 10 & 15 & 20 & 22 & 23 & 23 & 23 & 23 & \dots \\ 6 & 12 & 18 & 21 & 23 & 24 & 24 & 24 & 24 & \dots \\ 7 & 14 & 18 & 21 & 23 & 24 & 24 & 24 & 24 & \dots \\ 8 & 14 & 18 & 21 & 23 & 24 & 24 & 24 & 24 & \dots \\ 8 & 14 & 18 & 21 & 23 & 24 & 24 & 24 & 24 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is well known that if \mathbb{Y} is a fat point scheme in \mathbb{P}^n , then there exists a linear form that is a non-zerodivisor for the homogeneous coordinate ring of \mathbb{Y} . Using this property, the following lemma for fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ is shown in [8, Lemma 1.2].

Lemma 2.5. *Let \mathbb{Y} be a fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$. Then there exists a bihomogeneous element $L_1 \in S$ (respectively, $L_2 \in S$) with $\deg(L_1) = (1, 0)$ (respectively, $\deg(L_2) = (0, 1)$) such that $\overline{L_1}$ (respectively, $\overline{L_2}$) is a non-zerodivisor of $R_{\mathbb{Y}}$.*

Now we introduce a special class of fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.6. A fat point scheme \mathbb{Y} of $\mathbb{P}^1 \times \mathbb{P}^1$ is called *arithmetically Cohen-Macaulay* (or ACM for short) if its bihomogeneous coordinate ring $R_{\mathbb{Y}}$ is a Cohen-Macaulay ring.

In the case that \mathbb{Y} is an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, we can find a regular sequence of length two in $R_{\mathbb{Y}}$ of the following form (see [8, Theorem 4.4]).

Theorem 2.7. *Suppose that \mathbb{Y} is an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. Then there exist $L_1, L_2 \in S$ such that $\deg(L_1) = (1, 0)$ and $\deg(L_2) = (0, 1)$, and such that L_1, L_2 give rise to a regular sequence in $R_{\mathbb{Y}}$.*

Given a set of distinct points \mathbb{X} in $\mathbb{P}^1 \times \mathbb{P}^1$, and any point $P \in \mathbb{X}$, we sometimes want to compare the properties of \mathbb{X} with those of $\mathbb{X} \setminus \{P\}$. A separator gives us a tool to compare and contrast these two sets of points.

Definition 2.8. Let \mathbb{X} be a set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$ and let $(i, j) \in D_{\mathbb{X}}$. A bihomogeneous element $F \in S$ is called a *separator* of P_{ij} if $F(P_{ij}) \neq 0$, but $F(P_{kl}) = 0$ for all $(k, l) \in D_{\mathbb{X}} \setminus \{(i, j)\}$. We call F a *minimal separator* of P_{ij} if there does not exist a separator G of P_{ij} with $\deg G \prec \deg F$. In this case we also call $\overline{F} \in R_{\mathbb{X}}$ a minimal separator of P_{ij} .

Remark 2.9. Note that if $F \in S$ is a separator of a point P_{ij} of a set of distinct points \mathbb{X} in $\mathbb{P}^1 \times \mathbb{P}^1$, then the ideal $I_{\mathbb{X}} + \langle F \rangle$ is the vanishing ideal of the set $\mathbb{X} \setminus \{P_{ij}\}$.

More generally, let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ (with support $\mathbb{X} = \text{Supp}(\mathbb{Y})$), and let \wp_{ij} be the vanishing ideal of P_{ij} for every $(i, j) \in D_{\mathbb{X}}$.

Definition 2.10. Let $(i, j) \in D_{\mathbb{X}}$, and let $G \in S$ be a bihomogeneous element such that $G \in \wp_{ij}^{m_{ij}-1} \setminus \wp_{ij}^{m_{ij}}$ and $G \in \wp_{kl}^{m_{kl}}$ for all $(k, l) \in D_{\mathbb{X}} \setminus \{(i, j)\}$. Then G is called a *separator* of P_{ij} in \mathbb{Y} .

In this setting the preceding remark generalizes as follows.

Proposition 2.11. *Let \mathbb{Y} be a fat point subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ as above, let $(i, j) \in D_{\mathbb{X}}$, and let $G \in S$ be a separator of P_{ij} . Then the ideal $I_{\mathbb{Y}} + \langle G \rangle$ defines a subscheme of \mathbb{Y} of degree $\deg(\mathbb{Y}) - 1$.*

Proof. It suffices show that $I_{\mathbb{Y}} : \langle G \rangle = \wp_{ij}$ and to apply the short exact sequence

$$0 \longrightarrow S/(I_{\mathbb{Y}} : \langle G \rangle)(-\deg(G)) \xrightarrow{\times G} S/I_{\mathbb{Y}} \longrightarrow S/(I_{\mathbb{Y}} + \langle G \rangle) \longrightarrow 0.$$

Clearly, we have $\wp_{ij} \subseteq I_{\mathbb{Y}} : \langle G \rangle$. For the other inclusion, let $H \in I_{\mathbb{Y}} : \langle G \rangle$ be a bihomogeneous element. Then $GH \in I_{\mathbb{Y}} \subseteq \wp_{ij}^{m_{ij}}$. Using a change of coordinates, we may assume that $P_{ij} = [1 : 0] \times [1 : 0]$, and so $\wp_{ij} = \langle X_1, Y_1 \rangle$. We write $H = aX_0^k Y_0^l + H'$, where $H' \in \wp_{ij}$ with $\deg(H') = \deg(H) = (k, l)$ and $a \in K$. We get $GH = aX_0^k Y_0^l G + H'G$, and so $aX_0^k Y_0^l G \in \wp_{ij}^{m_{ij}}$. Since $G \notin \wp_{ij}^{m_{ij}}$, Macaulay's Basis Theorem (cf. [18, Theorem 1.5.7]) implies $a = 0$, and hence $H = H' \in \wp_{ij}$. \square

Next we present the notion of a set of minimal separators of a fat point in a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ which was introduced in [12].

Definition 2.12. Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, let $(i, j) \in D_{\mathbb{X}}$, and let $\mathbb{Y}' = \sum_{(k,l) \in D_{\mathbb{X}} \setminus \{(i,j)\}} m_{kl} P_{kl} + (m_{ij} - 1)P_{ij}$ be the fat point scheme obtained by reducing the multiplicity of P_{ij} by one. (If $m_{ij} = 1$, then the point P_{ij} does not appear in the support of \mathbb{Y}' .)

- a) A set $\{F_1, \dots, F_t\}$ is called a set of *minimal separators* of P_{ij} in \mathbb{Y} if $I_{\mathbb{Y}'} / I_{\mathbb{Y}} = \langle \overline{F}_1, \dots, \overline{F}_t \rangle$ and if there does not exist a set $\{G_1, \dots, G_u\}$ with $u < t$ such that $I_{\mathbb{Y}'} / I_{\mathbb{Y}} = \langle \overline{G}_1, \dots, \overline{G}_u \rangle$. In this case we also say that $\{\overline{F}_1, \dots, \overline{F}_t\} \subseteq R_{\mathbb{Y}}$ is a set of minimal separators of P_{ij} in \mathbb{Y} .
- b) The *degree tuple* of a minimal separators of P_{ij} in \mathbb{Y} is the tuple

$$\deg_{\mathbb{Y}}(P_{ij}) = (\deg(F_1), \dots, \deg(F_t))$$

where $\{F_1, \dots, F_t\}$ is any set of minimal separators of P_{ij} in \mathbb{Y} , relabeled such that $\deg(F_1) \leq_{\text{Lex}} \dots \leq_{\text{Lex}} \deg(F_t)$.

Note that the definition of $\deg_{\mathbb{Y}}(P_{ij})$ does not depend on the choice of (F_1, \dots, F_t) by [12, Theorem 3.6]. Furthermore, we deduce from [12, Theorem 6.4] and [11, Theorem 4.6] the following property.

Theorem 2.13. *Let \mathbb{Y}, \mathbb{Y}' be fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ as in Definition 2.12, and let $\deg_{\mathbb{Y}}(P_{ij}) = (d_1, \dots, d_t)$. Suppose that \mathbb{Y} is ACM, so that the minimal \mathbb{N}^2 -graded free resolution of $S/I_{\mathbb{Y}}$ has the form*

$$0 \rightarrow \mathbb{F}_2 = \bigoplus_{(k,l) \in \Sigma} S(-k, -l) \rightarrow \mathbb{F}_1 \rightarrow S \rightarrow S/I_{\mathbb{Y}} \rightarrow 0.$$

with a finite set $\Sigma \subset \mathbb{N}^2$. If \mathbb{Y}' is ACM then we have $t = m_{ij}$ and $d_k + (1, 1) \in \Sigma$ for $k = 1, \dots, m_{ij}$.

In particular, if \mathbb{Y} is an ACM set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, then the subset $\mathbb{Y}' = \mathbb{Y} \setminus \{P_{ij}\}$ is ACM if and only if $\deg_{\mathbb{Y}}(P_{ij}) + (1, 1) \in \Sigma$.

We end this section with a formula to compute the degree tuple of each fat point in an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ (see [14, Theorem 3.4]).

Theorem 2.14. *Let $\mathbb{Y} = \sum_{(k,l) \in D_{\mathbb{X}}} m_{kl} P_{kl}$ be an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, let $(i, j) \in D_{\mathbb{X}}$. For every $\ell \in \{0, \dots, m_{ij} - 1\}$, we set*

$$a_{\ell} = \sum_{(e,j) \in D_{\mathbb{X}}} \max\{m_{ej} - \ell, 0\} \quad \text{and} \quad b_{\ell} = \sum_{(i,p) \in D_{\mathbb{X}}} \max\{m_{ip} - \ell, 0\}.$$

Then we have

$$\deg_{\mathbb{Y}}(P_{ij}) = ((a_{m_{ij}-1} - 1, b_0 - 1), (a_{m_{ij}-2} - 1, b_1 - 1), \dots, (a_0 - 1, b_{m_{ij}-1} - 1)).$$

3. A PRESENTATION OF THE KÄHLER DIFFERENTIAL MODULE

In the following we let \mathbb{Y} be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ supported at \mathbb{X} . We also denote the image of X_i (respectively, Y_i) in $R_{\mathbb{Y}}$ by x_i (respectively, y_i) for $i = 0, 1$. Let L_1 and L_2 be two bihomogeneous elements given in Lemma 2.5. By making a linear change of coordinates in the X_i and a linear change of coordinates in the Y_i , we may assume that $L_1 = X_0$ and $L_2 = Y_0$, and that x_0, y_0 are non-zerodivisors of $R_{\mathbb{Y}}$.

In the bigraded algebra

$$R_{\mathbb{Y}} \otimes_K R_{\mathbb{Y}} = \bigoplus_{(i,j) \in \mathbb{N}^2} \left(\bigoplus_{\substack{i_1+i_2=i \\ j_1+j_2=j}} (R_{\mathbb{Y}})_{i_1, j_1} \otimes (R_{\mathbb{Y}})_{i_2, j_2} \right)$$

we have the bihomogeneous ideal $J = \ker(\mu)$, where $\mu : R_{\mathbb{Y}} \otimes_K R_{\mathbb{Y}} \rightarrow R_{\mathbb{Y}}$ is the bihomogeneous $R_{\mathbb{Y}}$ -linear map given by $\mu(f \otimes g) = fg$. Notice that

$$J = \langle x_i \otimes 1 - 1 \otimes x_i, y_i \otimes 1 - 1 \otimes y_i \mid i = 0, 1 \rangle.$$

Definition 3.1. The bigraded $R_{\mathbb{Y}}$ -module $\Omega_{R_{\mathbb{Y}}/K}^1 = J/J^2$ is called the *module of Kähler differentials* of $R_{\mathbb{Y}}/K$. The bihomogeneous K -linear map $d_{R_{\mathbb{Y}}/K} : R_{\mathbb{Y}} \rightarrow \Omega_{R_{\mathbb{Y}}/K}^1$ given by $f \mapsto f \otimes 1 - 1 \otimes f + J^2$ is called the *universal derivation* of $R_{\mathbb{Y}}/K$.

We collect some useful properties of the module of Kähler differentials. For a proof of these properties, see [20, Propositions 4.12 and 4.13].

Proposition 3.2. (i) *There is an isomorphism of bigraded $R_{\mathbb{Y}}$ -modules*

$$\Omega_{R_{\mathbb{Y}}/K}^1 \cong \Omega_{S/K}^1 / (dI_{\mathbb{Y}} + I_{\mathbb{Y}}\Omega_{S/K}^1)$$

where $\deg(dX_i) = (1, 0)$ and $\deg(dY_i) = (0, 1)$, and where

$$dI_{\mathbb{Y}} = \langle \frac{\partial F}{\partial X_0} dX_0 + \frac{\partial F}{\partial X_1} dX_1 + \frac{\partial F}{\partial Y_0} dY_0 + \frac{\partial F}{\partial Y_1} dY_1 \mid F \in I_{\mathbb{Y}} \rangle.$$

(ii) *The elements $\{dx_0, dx_1, dy_0, dy_1\}$ form a bihomogeneous system of generators of the bigraded $R_{\mathbb{Y}}$ -module $\Omega_{R_{\mathbb{Y}}/K}^1$.*

Remark 3.3. a) The bigraded S -module $\Omega_{S/K}^1$ has the representation

$$\Omega_{S/K}^1 = SdX_0 \oplus SdX_1 \oplus SdY_0 \oplus SdY_1 \cong S^2(-1, 0) \oplus S^2(0, -1).$$

b) If I_1, I_2 are bihomogeneous ideals of S , then

$$I_1\Omega_{S/K}^1 \cap I_2\Omega_{S/K}^1 = (I_1 \cap I_2)\Omega_{S/K}^1$$

(see [21, Chapter 3, §7, Theorem 7.4(i)]).

For convenience, we introduce the following notion.

Definition 3.4. Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. The fat point scheme $\mathbb{V} = \sum_{(i,j) \in D_{\mathbb{X}}} (m_{ij} + 1) P_{ij}$ is called the *thickening* of \mathbb{Y} .

Analogous to [16, Theorem 1.7], we have the following presentation of the module of Kähler differentials of $R_{\mathbb{Y}}/K$ when \mathbb{Y} is a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 3.5. *Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, and let \mathbb{V} be the thickening of \mathbb{Y} . Then the sequence of bigraded $R_{\mathbb{Y}}$ -modules*

$$0 \longrightarrow I_{\mathbb{Y}}/I_{\mathbb{V}} \longrightarrow R_{\mathbb{Y}}^2(-1, 0) \oplus R_{\mathbb{Y}}^2(0, -1) \longrightarrow \Omega_{R_{\mathbb{Y}}/K}^1 \longrightarrow 0$$

is exact.

In the proof of this theorem we use the following lemma.

Lemma 3.6. *In the setting of Theorem 3.5, let $\varphi : I_{\mathbb{Y}}/I_{\mathbb{V}} \rightarrow \Omega_{S/K}^1/I_{\mathbb{Y}}\Omega_{S/K}^1$ be the map given by $\varphi(F + I_{\mathbb{V}}) = dF + I_{\mathbb{Y}}\Omega_{S/K}^1$ for all $F \in I_{\mathbb{Y}}$. Then the map φ is well-defined, bihomogeneous of degree $(0, 0)$, and $R_{\mathbb{Y}}$ -linear.*

Proof. For $(i, j) \in D_{\mathbb{X}}$ we let \wp_{ij} be the associated ideal of $P_{ij} \in \mathbb{X} = \text{Supp}(\mathbb{Y})$. First we check that the map φ is well defined. Let $F_1, F_2 \in I_{\mathbb{Y}}$ be such that $F_1 - F_2 \in I_{\mathbb{V}}$. Then, for every $P_{ij} = Q_i \times R_j \in \mathbb{X}$, we have $F_1 - F_2 \in \wp_{ij}^{m_{ij}+1}$. Since $\wp_{ij} = \langle L_{Q_i}, L_{R_j} \rangle$, we have $\wp_{ij}^{m_{ij}+1} = \langle L_{Q_i}^{m_{ij}+1}, L_{Q_i}^{m_{ij}} L_{R_j}, \dots, L_{R_j}^{m_{ij}+1} \rangle$. So, we get

$$dF_1 - dF_2 \in d(\wp_{ij}^{m_{ij}+1}) \subset \langle L_{Q_i}^{m_{ij}}, L_{Q_i}^{m_{ij}-1} L_{R_j}, \dots, L_{R_j}^{m_{ij}} \rangle \Omega_{S/K}^1 = \wp_{ij}^{m_{ij}} \Omega_{S/K}^1.$$

Hence we see that

$$dF_1 - dF_2 \in \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}} \Omega_{S/K}^1.$$

Because $\Omega_{S/K}^1$ is a free S -module of rank 4, Remark 3.3 yields that

$$dF_1 - dF_2 \in \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}} \Omega_{S/K}^1 = I_{\mathbb{Y}} \Omega_{S/K}^1.$$

Moreover, it is clearly true that the map φ is bihomogeneous of degree $(0, 0)$. Now we prove that the map φ is $R_{\mathbb{Y}}$ -linear. For $F_1, F_2 \in I_{\mathbb{Y}}$ and $G_1, G_2 \in S$, we have

$$\begin{aligned} \varphi(G_1 F_1 + G_2 F_2 + I_{\mathbb{Y}}) &= d(G_1 F_1 + G_2 F_2) + I_{\mathbb{Y}} \Omega_{S/K}^1 \\ &= G_1 dF_1 + G_2 dF_2 + I_{\mathbb{Y}} \Omega_{S/K}^1 \\ &= (G_1 + I_{\mathbb{Y}}) \cdot \varphi(F_1 + I_{\mathbb{Y}}) + (G_2 + I_{\mathbb{Y}}) \cdot \varphi(F_2 + I_{\mathbb{Y}}). \end{aligned}$$

Hence the conclusion follows. \square

Proof of Theorem 3.5. Let $\varphi : I_{\mathbb{Y}}/I_{\mathbb{V}} \rightarrow \Omega_{S/K}^1/I_{\mathbb{Y}}\Omega_{S/K}^1$ be the $R_{\mathbb{Y}}$ -linear map given in Lemma 3.6. We see that

$$\text{im}(\varphi) = \varphi(I_{\mathbb{Y}}/I_{\mathbb{V}}) = (dI_{\mathbb{Y}} + I_{\mathbb{Y}}\Omega_{S/K}^1)/I_{\mathbb{Y}}\Omega_{S/K}^1,$$

and so

$$(\Omega_{S/K}^1/I_{\mathbb{Y}}\Omega_{S/K}^1)/\text{im}(\varphi) \cong \Omega_{S/K}^1/(dI_{\mathbb{Y}} + I_{\mathbb{Y}}\Omega_{S/K}^1) \cong \Omega_{R_{\mathbb{Y}}/K}^1.$$

In addition, we have

$$R_{\mathbb{Y}}^2(-1, 0) \oplus R_{\mathbb{Y}}^2(0, -1) \cong \Omega_{S/K}^1/I_{\mathbb{Y}}\Omega_{S/K}^1.$$

Hence it is enough to prove that the $R_{\mathbb{Y}}$ -linear map φ is an injection. To this end, let $F \in I_{\mathbb{Y}} \setminus I_{\mathbb{V}}$ be a bihomogeneous form of degree $(k, l) \in \mathbb{N}^2$. We need to prove $\varphi(F) = dF + I_{\mathbb{Y}}\Omega_{S/K}^1 \neq 0$ or $dF \notin I_{\mathbb{Y}}\Omega_{S/K}^1$. Since $I_{\mathbb{Y}}\Omega_{S/K}^1 = \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}} \Omega_{S/K}^1$, it suffices to show that $dF \notin \wp_{ij}^{m_{ij}} \Omega_{S/K}^1$ for some $(i, j) \in D_{\mathbb{X}}$.

We have $I_{\mathbb{Y}} = \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}}$ and $I_{\mathbb{V}} = \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}+1}$, and hence it follows from $F \in I_{\mathbb{Y}} \setminus I_{\mathbb{V}}$ that there is a point $P_{i_0, j_0} \in \mathbb{X}$ such that $F \in \wp_{i_0 j_0}^{m_{i_0 j_0}}$, but $F \notin \wp_{i_0 j_0}^{m_{i_0 j_0}+1}$. Since x_0, y_0 are non-zerodivisors of $R_{\mathbb{Y}}$, w.l.o.g. we may assume that

$$P_{i_0 j_0} = P = [1 : 0] \times [1 : 0] \in \mathbb{P}^1 \times \mathbb{P}^1.$$

Set $m = m_{i_0 j_0}$ and write $\wp_{i_0 j_0} = \wp = \langle X_1^i Y_1^j \mid i + j = t \rangle$ for all $t \geq 1$. Since $F \in \wp^m \setminus \wp^{m+1}$ and $\deg(F) = (k, l)$, we have the following representation of F :

$$F = \sum_{\substack{i+j=m+1 \\ (i,j) \preceq (k,l)}} X_1^i Y_1^j F_{i,j} + \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} X_1^i Y_1^j G_{i,j}$$

where $F_{i,j} \in S$ with $i + j = m + 1$ and $\deg(F_{i,j}) = (k - i, l - j)$, and where $G_{i,j} \in K[X_0, Y_0]$ with $i + j = m$ and $\deg(G_{i,j}) = (k - i, l - j)$. We write $G_{i,j} = a_{i,j} X_0^{k-i} Y_0^{l-j}$ with $a_{i,j} \in K$. Since $F \notin \wp^{m+1}$, not all elements $a_{i,j}$ are zero. Setting

$\tilde{F} = \sum_{i+j=m+1, (i,j) \preceq (k,l)} X_1^i Y_1^j F_{i,j}$, we have $d\tilde{F} \in \wp^m \Omega_{S/K}^1$ and

$$\begin{aligned} dF &= d\tilde{F} + d\left(\sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} X_1^i Y_1^j G_{i,j}\right) \\ &= d\tilde{F} + \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} d(a_{i,j} X_1^i Y_1^j X_0^{k-i} Y_0^{l-j}) \\ &= (d\tilde{F} + \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} a_{i,j} X_1^i Y_1^j d(X_0^{k-i} Y_0^{l-j})) + \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} a_{i,j} X_0^{k-i} Y_0^{l-j} d(X_1^i Y_1^j). \end{aligned}$$

Note that

$$d\tilde{F} + \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} a_{i,j} X_1^i Y_1^j d(X_0^{k-i} Y_0^{l-j}) \in \wp^m \Omega_{S/K}^1.$$

So, in order to prove $dF \notin \wp^m \Omega_{S/K}^1$, it suffices to prove that

$$w := \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} a_{i,j} X_0^{k-i} Y_0^{l-j} d(X_1^i Y_1^j) \notin \wp^m \Omega_{S/K}^1.$$

Observe that

$$\begin{aligned} w &= \sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} a_{i,j} X_0^{k-i} Y_0^{l-j} (i X_1^{i-1} Y_1^j dX_1 + j X_1^i Y_1^{j-1} dY_1) \\ &= \left(\sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} i a_{i,j} X_0^{k-i} Y_0^{l-j} X_1^{i-1} Y_1^j\right) dX_1 + \left(\sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} j a_{i,j} X_0^{k-i} Y_0^{l-j} X_1^i Y_1^{j-1}\right) dY_1. \end{aligned}$$

The ideal \wp^m is a monomial ideal of S generated by the set $\{X_1^i Y_1^j \mid i+j=m\}$. Macaulay's Basis Theorem yields that for all $(i,j) \in \mathbb{N}^2$ with $i+j=m$ and $(i,j) \preceq (k,l)$ we have $X_0^{k-i} Y_0^{l-j} X_1^{i-1} Y_1^j \notin \wp^m$ if $i > 0$ and $X_0^{k-i} Y_0^{l-j} X_1^i Y_1^{j-1} \notin \wp^m$ if $j > 0$, and that any non-zero polynomial which has support contained in the set of these terms does not belong to \wp^m . Because not all elements $a_{i,j}$ are zero and $i+j=m \geq 1$, it follows that

$$\left(\sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} i a_{i,j} X_0^{k-i} Y_0^{l-j} X_1^{i-1} Y_1^j\right) dX_1 \notin \wp^m \Omega_{S/K}^1$$

or

$$\left(\sum_{\substack{i+j=m \\ (i,j) \preceq (k,l)}} j a_{i,j} X_0^{k-i} Y_0^{l-j} X_1^i Y_1^{j-1}\right) dY_1 \notin \wp^m \Omega_{S/K}^1$$

Note that $\Omega_{S/K}^1$ is free S -module with basis $\{dX_0, dX_1, dY_0, dY_1\}$. So, we get $w \notin \wp^m \Omega_{S/K}^1$. Hence we have shown that $dF \notin \wp^m \Omega_{S/K}^1 = \wp_{i_0 j_0}^{m_{i_0 j_0}} \Omega_{S/K}^1$. This means $\varphi(F) \neq 0$ for any $F \in I_{\mathbb{Y}} \setminus I_{\mathbb{V}}$. Therefore φ is an injection, as wanted. \square

Using Theorem 3.5, we get a relation between the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ and of \mathbb{Y} and \mathbb{V} .

Corollary 3.7. *Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{K}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, and let \mathbb{V} be the thickening of \mathbb{Y} . Then the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ satisfies*

$$\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i,j) = 2 \mathrm{HF}_{\mathbb{Y}}(i-1,j) + 2 \mathrm{HF}_{\mathbb{Y}}(i,j-1) + \mathrm{HF}_{\mathbb{Y}}(i,j) - \mathrm{HF}_{\mathbb{V}}(i,j)$$

for all $(i,j) \in \mathbb{Z}^2$.

Proposition 3.8. *Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with support $\mathbb{X} = \text{Supp}(\mathbb{Y})$. The following statements hold true.*

- (i) $\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 0$ if $(i, j) \not\geq (0, 0)$ or $(i, j) = (0, 0)$.
- (ii) For $(i, j) \in \mathbb{N}^2$, if $\text{HF}_{I_{\mathbb{Y}}}(i, j) = 0$, then

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 4ij + 2i + 2j.$$

- (iii) If $\text{HF}_{\mathbb{Y}}(i_0 - 1, j_0) = \text{HF}_{\mathbb{Y}}(i_0, j_0) = \text{HF}_{\mathbb{Y}}(i_0, j_0 - 1)$, then for $(i, j) \succeq (i_0, j_0)$ we have

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 5 \text{HF}_{\mathbb{Y}}(i, j) - \text{HF}_{\mathbb{V}}(i, j)$$

where \mathbb{V} is the thickening of \mathbb{Y} .

Proof. Claim (i) follows from the representation of Kähler differential module

$$\Omega_{R_{\mathbb{Y}}/K}^1 = R_{\mathbb{Y}} dx_0 + R_{\mathbb{Y}} dx_1 + R_{\mathbb{Y}} dy_0 + R_{\mathbb{Y}} dy_1.$$

Observe that $I_{\mathbb{V}} \subseteq I_{\mathbb{Y}}$ where $\mathbb{V} = \sum_{(i,j) \in D_{\mathbb{X}}} (m_{ij} + 1) P_{ij}$. So, for $(i, j) \in \mathbb{N}^2$ such that $\text{HF}_{I_{\mathbb{Y}}}(i, j) = 0$ we have $\text{HF}_{I_{\mathbb{V}}}(i, j) = 0$ and $\text{HF}_{\mathbb{Y}}(i, j) = \text{HF}_{\mathbb{V}}(i, j) = (i + 1)(j + 1)$. Hence claim (ii) follows from Corollary 3.7. Finally, claim (iii) is a consequence of Corollary 3.7 and [8, Proposition 1.3]. \square

Example 3.9. Let $\mathbb{X} = \{P_{11}, P_{12}, P_{23}, P_{31}, P_{32}\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of distinct points as Example 2.4. Let \mathbb{Y} and \mathbb{V} be the two fat point schemes supported at \mathbb{X}

$$\mathbb{Y} = 2P_{11} + 2P_{12} + P_{23} + P_{31} + 2P_{32} \text{ and } \mathbb{V} = 3P_{11} + 3P_{12} + 2P_{23} + 2P_{31} + 3P_{32}.$$

Based on the definition of $\Omega_{R_{\mathbb{Y}}/K}^1$ or by applying Corollary 3.7, we compute the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1} = \begin{bmatrix} 0 & 2 & 4 & 6 & 8 & 9 & 8 & 7 & 7 & \dots \\ 2 & 8 & 14 & 20 & 24 & 22 & 20 & 20 & 20 & \dots \\ 4 & 14 & 24 & 32 & 31 & 28 & 28 & 28 & 28 & \dots \\ 6 & 20 & 32 & 35 & 33 & 32 & 32 & 32 & 32 & \dots \\ 8 & 24 & 31 & 33 & 33 & 32 & 32 & 32 & 32 & \dots \\ 9 & 22 & 28 & 32 & 32 & 31 & 31 & 31 & 31 & \dots \\ 8 & 20 & 28 & 32 & 32 & 31 & 31 & 31 & 31 & \dots \\ 7 & 20 & 28 & 32 & 32 & 31 & 31 & 31 & 31 & \dots \\ 7 & 20 & 28 & 32 & 32 & 31 & 31 & 31 & 31 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Clearly, $\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(0, 0) = 0$. If $(i, j) = (2, 2)$ then $(I_{\mathbb{Y}})_{(2,2)} = \langle 0 \rangle$. We see that $\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(2, 2) = 24 = 4ij + 2i + 2j$. For $(i, j) \succeq (4, 4)$, we also see that $\text{HF}_{\mathbb{Y}}(3, 4) = \text{HF}_{\mathbb{Y}}(4, 4) = \text{HF}_{\mathbb{Y}}(4, 3)$, and hence $\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 5 \text{HF}_{\mathbb{Y}}(i, j) - \text{HF}_{\mathbb{V}}(i, j)$. In particular, if $(i, j) \succeq (5, 5)$ then $\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 31 = 5 \text{HF}_{\mathbb{Y}}(i, j) - \text{HF}_{\mathbb{V}}(i, j)$.

4. THE HILBERT FUNCTION OF THE KÄHLER DIFFERENTIAL MODULE

We continue to use the notation introduced in Section 2. In [8], giving a set of fat points \mathbb{Y} , the authors associated to \mathbb{Y} two tuples $\alpha_{\mathbb{Y}}$ and $\beta_{\mathbb{Y}}$ to describe all but finitely many values of the Hilbert function of $R_{\mathbb{Y}}$. In this section we show that the same tuples enable us to describe all but a finite number of values of the Hilbert function of the module of Kähler differentials of a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projection morphisms given by $P_1 \times P_2 \mapsto P_1$ and $P_1 \times P_2 \mapsto P_2$, respectively. We recall from [8] the following notation and definitions.

Notation 4.1. Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with support $\mathbb{X} = \text{Supp}(\mathbb{Y})$, and let $\pi_1(\mathbb{X}) = \{Q_1, \dots, Q_r\}$ and $\pi_2(\mathbb{X}) = \{R_1, \dots, R_t\}$.

- a) For each $Q_i \in \pi_1(\mathbb{X})$, we define $\mathbb{Y}_{1,Q_i} := m_{ij_1} P_{ij_1} + \dots + m_{ij_{\nu_i}} P_{ij_{\nu_i}}$ where $P_{ij_k} = Q_i \times R_{j_k}$ are those points of \mathbb{X} whose first coordinate is Q_i . Set

$$l_i := \max\{m_{ij_1}, \dots, m_{ij_{\nu_i}}\} \quad \text{and} \quad \alpha_{Q_i} := (a_{i0}, \dots, a_{il_i-1})$$

where $a_{ik} := \sum_{e=1}^{\nu_i} \max\{m_{ije} - k, 0\}$ with $0 \leq k \leq l_i - 1$. By rearranging the elements of $(\alpha_{Q_1}, \dots, \alpha_{Q_r})$ in non-increasing order, we get the $(l_1 + \dots + l_r)$ -tuple which is denoted by $\alpha_{\mathbb{Y}}$.

- b) Suppose $\alpha_{\mathbb{Y}} = (\alpha_1, \dots, \alpha_l)$ where $l = \sum_{i=1}^r l_i$. We define the *conjugate* of $\alpha_{\mathbb{Y}}$

$$\alpha_{\mathbb{Y}}^* = (\alpha_1^*, \alpha_2^*, \dots) \quad \text{with} \quad \alpha_i^* = \#\{\alpha_j \in \alpha_{\mathbb{Y}} \mid \alpha_j \geq i\}.$$

Here $\alpha_i^* = 0$ for $i > \alpha_1$.

- c) Similarly, for every $R_j \in \pi_2(\mathbb{X})$, we set $\mathbb{Y}_{2,R_j} := m_{i_1j} P_{i_1j} + \dots + m_{i_{\nu'_j}j} P_{i_{\nu'_j}j}$ where $P_{i_kj} = Q_{i_k} \times R_j$ are those points of \mathbb{X} whose second coordinate is R_j . We also set

$$l'_j := \max\{m_{i_1j}, \dots, m_{i_{\nu'_j}j}\} \quad \text{and} \quad \beta_{R_j} := (b_{j0}, \dots, b_{jl'_j-1})$$

where $b_{jk} := \sum_{e=1}^{\nu'_j} \max\{m_{ie_j} - k, 0\}$ with $0 \leq k \leq l'_j - 1$. We let $\beta_{\mathbb{Y}}$ denote the $(l'_1 + \dots + l'_t)$ -tuple which is obtained by rearranging the elements of $(\beta_{R_1}, \dots, \beta_{R_t})$ in non-increasing order.

- d) Suppose that $\beta_{\mathbb{Y}} = (\beta_1, \dots, \beta_{l'})$ where $l' = \sum_{j=1}^t l'_j$. We define the *conjugate* of $\beta_{\mathbb{Y}}$

$$\beta_{\mathbb{Y}}^* = (\beta_1^*, \beta_2^*, \dots) \quad \text{with} \quad \beta_j^* = \#\{\beta_k \in \beta_{\mathbb{Y}} \mid \beta_k \geq j\}.$$

Note that here it is $\beta_j^* = 0$ for $j > \beta_1$.

With the above notation, we also define:

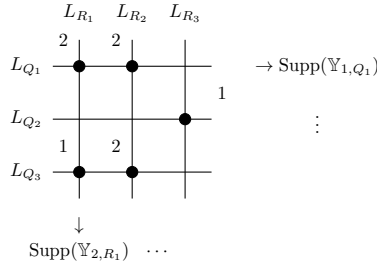
$$\hat{\alpha}_{\mathbb{Y}} := (a_{10} + \nu_1, \dots, a_{r0} + \nu_r)$$

and

$$\hat{\beta}_{\mathbb{Y}} := (b_{10} + \nu'_1, \dots, b_{t0} + \nu'_t).$$

Let us calculate the tuples of Notation 4.1 in a concrete case.

Example 4.2. Let $\mathbb{Y} = 2P_{11} + 2P_{12} + P_{23} + P_{31} + 2P_{32}$ be the fat point scheme of Example 3.9. We want to compute the tuples $\alpha_{\mathbb{Y}}$, $\hat{\alpha}_{\mathbb{Y}}$, $\alpha_{\mathbb{Y}}^*$, $\beta_{\mathbb{Y}}$, $\hat{\beta}_{\mathbb{Y}}$ and $\beta_{\mathbb{Y}}^*$. We have $\mathbb{X} = \text{Supp}(\mathbb{Y})$, $\pi_1(\mathbb{X}) = \{Q_1, Q_2, Q_3\}$ and $\pi_2(\mathbb{X}) = \{R_1, R_2, R_3\}$. Also, we have $\mathbb{Y}_{1,Q_1} = 2P_{11} + 2P_{12}$. Note that the support of \mathbb{Y}_{1,Q_1} contains the points of \mathbb{X} which lie on the horizontal ruling defined by L_{Q_1} (see the figure).



We set $l_1 = \max\{m_{11}, m_{12}\} = 2$, $l_2 = m_{21} = 1$, and $l_3 = \max\{m_{31}, m_{32}\} = 2$. Then

$$\begin{aligned} a_{10} &= \sum_{e=1}^2 \max\{m_{1e} - 0, 0\} = 4, & a_{11} &= \sum_{e=1}^2 \max\{m_{1e} - 1, 0\} = 2, \\ a_{20} &= \sum_{e=1}^2 \max\{m_{2e} - 0, 0\} = 1, & a_{21} &= \sum_{e=1}^2 \max\{m_{2e} - 1, 0\} = 0, \\ a_{30} &= \sum_{e=1}^2 \max\{m_{3e} - 0, 0\} = 3, & a_{31} &= \sum_{e=1}^2 \max\{m_{3e} - 1, 0\} = 1. \end{aligned}$$

So, we have $\alpha_{Q_1} = (4, 2)$, $\alpha_{Q_2} = (1)$ and $\alpha_{Q_3} = (3, 1)$. Thus we obtain $\alpha_{\mathbb{Y}} = (4, 3, 2, 1, 1)$ and $\hat{\alpha}_{\mathbb{Y}} = (a_{10} + 2, a_{20} + 1, a_{30} + 2) = (6, 2, 5)$. Moreover, it follows that $\alpha_{\mathbb{Y}}^* = (5, 3, 2, 1, 0, 0, \dots)$.

Similarly, for $R_1, R_2, R_3 \in \pi_2(\mathbb{X})$, we find $l'_1 = 2$, $l'_2 = 2$ and $l'_3 = 1$, respectively. Also, we have $\beta_{R_1} = (3, 1)$, $\beta_{R_2} = (4, 2)$ and $\beta_{R_3} = (1)$. Hence we get $\beta_{\mathbb{Y}} = (4, 3, 2, 1, 1)$ and $\hat{\beta}_{\mathbb{Y}} = (5, 6, 2)$. Also, we have $\beta_{\mathbb{Y}}^* = (5, 3, 2, 1, 0, 0, \dots)$.

Using these notations, we can give a formula for the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$, as the following proposition shows.

Proposition 4.3. *Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with associated tuples $\hat{\alpha}_{\mathbb{Y}}$, $\alpha_{\mathbb{Y}}^*$, $\hat{\beta}_{\mathbb{Y}}$, and $\beta_{\mathbb{Y}}^*$ as given in Notation 4.1.*

- (i) *For all $j \in \mathbb{N}$, if $i \geq l + r - 1$ and let $h = \min\{j + 1, \max\{a \in \hat{\alpha}_{\mathbb{Y}}\}\}$, then we have*

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 4 \sum_{k=1}^j \alpha_k^* + 2\alpha_{j+1}^* - \sum_{k=1}^h \#\{a \in \hat{\alpha}_{\mathbb{Y}} \mid a \geq k\}.$$

- (ii) *For all $i \in \mathbb{N}$, if $j \geq l' + t - 1$ and let $h' = \min\{i + 1, \max\{b \in \hat{\beta}_{\mathbb{Y}}\}\}$, then we have*

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 4 \sum_{k=1}^i \beta_k^* + 2\beta_{i+1}^* - \sum_{k=1}^{h'} \#\{b \in \hat{\beta}_{\mathbb{Y}} \mid b \geq k\}.$$

Proof. We need only prove claim (i) because claim (ii) follows in the same way. Note that $\mathbb{X} = \text{Supp}(\mathbb{Y})$. For every $Q_i \in \pi_1(\mathbb{X})$, we write $\alpha_{Q_i} = (a_{i0}, \dots, a_{i l_i - 1})$ as given in Notation 4.1.a. Let $\mathbb{V} = \sum_{(i,j) \in D_{\mathbb{X}}} (m_{ij} + 1) P_{ij}$ be the thickening of \mathbb{Y} and $\tilde{\alpha}_{\mathbb{V}} := (\alpha'_{Q_1}, \dots, \alpha'_{Q_r})$ the tuple associated to \mathbb{V} where $\alpha'_{Q_i} = (a'_{i0}, \dots, a'_{i l_i})$. We see that

$$a'_{i0} = a_{i0} + \nu_i \text{ and } a'_{ik} = \sum_{e=1}^{\nu_i} \max\{m_{ije} + 1 - k, 0\} = a_{ik-1}$$

for $k = 1, \dots, l_i$, and hence $\alpha'_{Q_i} = (a_{i0} + \nu_i, a_{i0}, \dots, a_{il_i-1})$. Consequently, if $\tilde{\alpha}_{\mathbb{Y}} := (\alpha_{Q_1}, \dots, \alpha_{Q_r})$ is the tuple associated to \mathbb{Y} , and in a non-increasing ordering we order all the elements of the union $\{a \mid a \in \tilde{\alpha}_{\mathbb{Y}}\} \cup \{a_{10} + \nu_1, \dots, a_{r0} + \nu_r\}$, we get the tuple $\alpha_{\mathbb{V}}$ associated to \mathbb{V} . Consider $\alpha_{\mathbb{Y}} = (\alpha_1, \dots, \alpha_l)$, $\alpha_{\mathbb{Y}}^* = (\alpha_1^*, \dots, \alpha_{\alpha_1}^*)$, and $\alpha_{\mathbb{V}} = (\alpha'_1, \dots, \alpha'_{l+r})$.

For $i = 1, \dots, \alpha_1$, we observe that

$$\begin{aligned} (\alpha'_i)^* &= \#\{\alpha'_j \in \alpha_{\mathbb{V}} \mid \alpha'_j \geq i\} \\ &= \#\{\alpha_j \in \alpha_{\mathbb{Y}} \mid \alpha_j \geq i\} + \#\{a \in \hat{\alpha}_{\mathbb{Y}} \mid a \geq i\} \\ &= \alpha_i^* + \#\{a \in \hat{\alpha}_{\mathbb{Y}} \mid a \geq i\} \\ &= \alpha_i^* + \hat{\alpha}_i^*. \end{aligned}$$

If $\alpha_1 < i \leq \alpha'_1$ we have

$$(\alpha'_i)^* = \#\{\alpha'_j \in \alpha_{\mathbb{V}} \mid \alpha'_j \geq i\} = \#\{\hat{\alpha}_j \in \hat{\alpha}_{\mathbb{Y}} \mid \hat{\alpha}_j \geq i\} = \hat{\alpha}_i^*,$$

and if $i > \alpha'_1$ we have $(\alpha'_i)^* = 0$. Hence, by [8, Theorem 3.2], for $j \in \mathbb{N}$ and for $i \geq l + r - 1$ we have

$$\text{HF}_{\mathbb{V}}(i-1, j) = \sum_{k=1}^{j+1} \alpha_k^*$$

and

$$\text{HF}_{\mathbb{V}}(i, j) = \sum_{k=1}^{j+1} \alpha_k^* + \sum_{k=1}^h \#\{a \in \hat{\alpha}_{\mathbb{Y}} \mid a \geq k\} = \sum_{k=1}^{j+1} \alpha_k^* + \sum_{k=1}^h \hat{\alpha}_k^*$$

where $h = \min\{j+1, \max\{a \in \hat{\alpha}_{\mathbb{Y}}\}\} = \min\{j+1, \alpha'_1\}$. Moreover, Corollary 3.7 shows that

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 2 \text{HF}_{\mathbb{Y}}(i-1, j) + 2 \text{HF}_{\mathbb{Y}}(i, j-1) + \text{HF}_{\mathbb{Y}}(i, j) - \text{HF}_{\mathbb{V}}(i, j)$$

for all (i, j) . Therefore, for $j \in \mathbb{N}$ and for $i \geq l + r - 1$, the desired formula of $\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}$ follows. \square

Based on this proposition, we can work out certain values of the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ explicitly.

Corollary 4.4. *Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ and let \mathbb{V} be the thickening of \mathbb{Y} .*

(i) *For all $(i, j) \succeq (l + r - 1, l' + t - 1)$, we have*

$$\text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 4 \sum_{(i,j) \in D_{\mathbb{X}}} \binom{m_{ij} + 1}{2} - \sum_{(i,j) \in D_{\mathbb{X}}} (m_{ij} + 1).$$

(ii) *If $i \geq l + r - 1$ and $j < l' + t - 1$, then*

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) &= \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(l + r - 1, j) \\ &= 3 \text{HF}_{\mathbb{Y}}(l + r - 1, j) + 2 \text{HF}_{\mathbb{Y}}(l + r - 1, j - 1) - \text{HF}_{\mathbb{V}}(l + r - 1, j). \end{aligned}$$

(iii) *If $i < l + r - 1$ and $j \geq l' + t - 1$, then*

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) &= \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, l' + t - 1) \\ &= 2 \text{HF}_{\mathbb{Y}}(i - 1, l' + t - 1) + 3 \text{HF}_{\mathbb{Y}}(i, l' + t - 1) - \text{HF}_{\mathbb{V}}(i, l' + t - 1). \end{aligned}$$

(iv) For $(i, j) \succeq (l, l')$, we have

$$\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) \geq \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i+1, j) \text{ and } \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) \geq \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j+1).$$

Proof. According to Proposition 4.3, we have

$$\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = \begin{cases} \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(l+r-1, l'+t-1) & \text{if } (i, j) \succeq (l+r-1, l'+t-1), \\ \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(l+r-1, j) & \text{if } i \geq l+r-1, \\ \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, l'+t-1) & \text{if } j \geq l'+t-1. \end{cases}$$

So, Corollary 3.7 and [8, Corollary 3.4] imply claims (ii), (iii) and $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 5 \sum_{(i,j) \in D_{\mathbb{X}}} \binom{m_{ij}+1}{2} - \sum_{(i,j) \in D_{\mathbb{X}}} \binom{m_{ij}+2}{2}$ for all $(i, j) \succeq (l+r-1, l'+t-1)$. Thus claim (i) also follows. Furthermore, for $(i, j) \succeq (l, l')$ we see that $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 5 \mathrm{HF}_{\mathbb{Y}}(i, j) - \mathrm{HF}_{\mathbb{Y}}(i, j) = 5 \sum_{(i,j) \in D_{\mathbb{X}}} \binom{m_{ij}+1}{2} - \mathrm{HF}_{\mathbb{Y}}(i, j)$ by [8, Corollary 3.4]. We also have $\mathrm{HF}_{\mathbb{Y}}(i, j) \leq \mathrm{HF}_{\mathbb{Y}}(i+1, j)$ and $\mathrm{HF}_{\mathbb{Y}}(i, j) \leq \mathrm{HF}_{\mathbb{Y}}(i, j+1)$ for all $(i, j) \in \mathbb{N}^2$ by [8, Proposition 1.3(i)], and therefore claim (iv) holds true. \square

Remark 4.5. The above corollary tells us that if we know the values of $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(l+r-1, j)$ for $j = 0, \dots, l'+t$ and the values of $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, l'+t-1)$ for $i = 0, \dots, l+r$, then we know all but a finite numbers of values of the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$.

Example 4.6. Let us go back to Example 3.9. We see that the fat point scheme \mathbb{Y} satisfies $l = l' = 5$ and $r = t = 3$. For $(i, j) \succeq (7, 7)$, we have

$$\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = 31 = 4 \sum_{(i,j) \in D_{\mathbb{X}}} \binom{m_{ij}+1}{2} - \sum_{(i,j) \in D_{\mathbb{X}}} (m_{ij}+1).$$

Moreover, for $(i, j) \succeq (5, 5)$, we have $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) \geq \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i+1, j)$ and $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) \geq \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j+1)$.

5. SPECIAL FAT POINT SCHEMES IN $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we study cases in which the support \mathbb{X} of the fat point scheme \mathbb{Y} is either a complete intersection or an almost complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us begin with the complete intersection case.

Definition 5.1. Let I be an ideal in $S = K[X_0, X_1, Y_0, Y_1]$.

- The ideal I is called a *(bihomogeneous) complete intersection* if it is generated by a (bihomogeneous) regular sequence.
- A set of distinct points \mathbb{X} in $\mathbb{P}^1 \times \mathbb{P}^1$ is called a *complete intersection* if its vanishing ideal $I_{\mathbb{X}}$ is a bihomogeneous complete intersection.
- If the vanishing ideal of a set of distinct points \mathbb{X} in $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by two bihomogeneous polynomials of degrees $(d_1, 0)$ and $(0, d_2)$, where $d_1, d_2 \geq 1$, we say that \mathbb{X} is a complete intersection of type (d_1, d_2) and write $CI(d_1, d_2)$.

Remark 5.2. Note that \mathbb{X} is a complete intersection if and only if it is a $CI(d_1, d_2)$ for some $d_1, d_2 \geq 1$ (cf. [5, Theorem 1.2 and Remark 1.3]). Moreover, a complete intersection is arithmetically Cohen-Macaulay and [15, Lemma 2.26] provides an explicit description of the bigraded minimal free resolution of its bihomogeneous coordinate ring.

Now we can apply the results of the previous sections to compute the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ in the case that \mathbb{Y} is an equimultiple fat point scheme whose support \mathbb{X} is a complete intersection of type (d_1, d_2) . In the following, unless stated otherwise, we assume that \mathbb{X} is a set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$ which is a $CI(d_1, d_2)$ with $d_1 \leq d_2$. For more details about complete intersections see [15, Section 5.4].

The first easy example of a complete intersection is the case when \mathbb{X} is a $CI(1, d_2)$, i.e., when \mathbb{X} is a set of distinct points on a line. In the following example we present a formula for the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ depending only on the tuple $\alpha_{\mathbb{Y}}$ when \mathbb{Y} is a fat point scheme whose support \mathbb{X} is a set of s points on a line.

Example 5.3. Let $\mathbb{Y} = m_{11}P_{11} + m_{12}P_{12} + \cdots + m_{1s}P_{1s}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is contained in a line defined by a form of degree $(1, 0)$. Suppose that $\alpha_{\mathbb{Y}} = (a_1, \dots, a_l)$ with $l = \max\{m_{11}, \dots, m_{1s}\}$ is the tuple associated with \mathbb{Y} . Let \mathcal{Z}_k be a matrix of k rows and infinitely many columns with all entries equal to zero, and let

$$\mathcal{A}_k = \begin{bmatrix} 1 & 2 & \cdots & a_k - 1 & a_k & a_k & \cdots \\ 1 & 2 & \cdots & a_k - 1 & a_k & a_k & \cdots \\ 1 & 2 & \cdots & a_k - 1 & a_k & a_k & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then we have $\text{HF}_{\mathbb{Y}} = \sum_{k=1}^l \begin{bmatrix} \mathcal{Z}_{k-1} \\ \mathcal{A}_k \end{bmatrix}$ by [8, Theorem 2.2]. Note that the associated tuple of the scheme $\mathbb{V} = (m_{11} + 1)P_{11} + (m_{12} + 1)P_{12} + \cdots + (m_{1s} + 1)P_{1s}$ is given by $\alpha_{\mathbb{V}} = (a_1 + s, a_1, \dots, a_l)$. By letting

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & \cdots & a_1 + s - 1 & a_1 + s & a_1 + s & \cdots \\ 1 & 2 & \cdots & a_1 + s - 1 & a_1 + s & a_1 + s & \cdots \\ 1 & 2 & \cdots & a_1 + s - 1 & a_1 + s & a_1 + s & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

we have

$$\text{HF}_{\mathbb{V}} = \mathcal{A} + \sum_{k=1}^l \begin{bmatrix} \mathcal{Z}_k \\ \mathcal{A}_k \end{bmatrix}.$$

Now we let \mathcal{B}_k be the matrix obtained from \mathcal{A}_k by adding one zero column to \mathcal{A}_k in the first position. Notice that $\text{HF}_{\mathbb{Y}}(i, j) = 0$ if $(i, j) \not\prec (0, 0)$. An application of Corollary 3.7 yields

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1} &= \sum_{k=1}^l 2 \begin{bmatrix} \mathcal{Z}_k \\ \mathcal{A}_k \end{bmatrix} + \sum_{k=1}^l 2 \begin{bmatrix} \mathcal{Z}_{k-1} \\ \mathcal{B}_k \end{bmatrix} + \sum_{k=1}^l \begin{bmatrix} \mathcal{Z}_{k-1} \\ \mathcal{A}_k \end{bmatrix} - \mathcal{A} - \sum_{k=1}^l \begin{bmatrix} \mathcal{Z}_k \\ \mathcal{A}_k \end{bmatrix} \\ &= \sum_{k=1}^l \begin{bmatrix} \mathcal{Z}_k \\ \mathcal{A}_k \end{bmatrix} + \sum_{k=1}^l 2 \begin{bmatrix} \mathcal{Z}_{k-1} \\ \mathcal{B}_k \end{bmatrix} + \sum_{k=1}^l \begin{bmatrix} \mathcal{Z}_{k-1} \\ \mathcal{A}_k \end{bmatrix} - \mathcal{A}. \end{aligned}$$

It is well known (cf. [8, Corollary 4.9]) that a fat point scheme \mathbb{V} whose support is on a line is ACM. In this case we can characterize the Hilbert function of the Kähler differential module of \mathbb{Y} in terms of $\alpha_{\mathbb{Y}}$ and $\hat{\alpha}_{\mathbb{Y}}$ as the preceding example showed. When the thickening \mathbb{V} of \mathbb{Y} is an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, this characterization can be generalized as follows.

Proposition 5.4. *Let \mathbb{X} be a set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be a fat point scheme with associated tuples $\hat{\alpha}_{\mathbb{Y}}$ and $\alpha_{\mathbb{Y}}$, and let \mathbb{V} be the thickening of \mathbb{Y} . Suppose that \mathbb{V} is ACM and that $\hat{\alpha}_1 \geq \dots \geq \hat{\alpha}_r \geq \alpha_1$. Let $c_{ij} = \min\{j+1, \alpha_{i+1}\}$ for $i = 0, \dots, l-1$ and all $j \in \mathbb{N}$. Then the Hilbert function of $\Omega_{R_{\mathbb{Y}}/K}^1$ is*

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i, j) = & 2 \sum_{k=0}^{\min\{i-1, l-1\}} c_{kj} + 2 \sum_{k=0}^{\min\{i, l-1\}} c_{k, j-1} + \sum_{k=0}^{\min\{i, l-1\}} c_{kj} \\ & - \sum_{k=0}^{\min\{i, r-1\}} \min\{j+1, \hat{\alpha}_{k+1}\} - \sum_{k=r}^{\min\{i, l+r-1\}} c_{k-r, j} \end{aligned}$$

for all $(i, j) \in \mathbb{N}^2$.

Proof. We see that $\alpha_{\mathbb{V}} = (\hat{\alpha}_1, \dots, \hat{\alpha}_r, \alpha_1, \dots, \alpha_l)$. So, by Corollary 4.10 in [8], we have

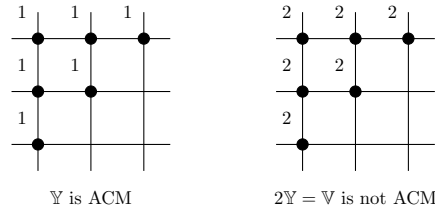
$$\mathrm{HF}_{\mathbb{V}}(i, j) = \sum_{k=0}^{\min\{i, r-1\}} \min\{j+1, \hat{\alpha}_{k+1}\} + \sum_{k=r}^{\min\{i, l+r-1\}} c_{k-r, j}.$$

Note that \mathbb{V} and \mathbb{Y} have the same support and \mathbb{V} is ACM, and so \mathbb{Y} is ACM by [8, Theorem 4.2], and by the fact that $\mathcal{S}_{\mathbb{Y}} \subseteq \mathcal{S}_{\mathbb{V}}$, where $\mathcal{S}_{\mathbb{Y}}$ is the set of t -tuples $\mathcal{S}_{\mathbb{Y}} = \{(\max\{0, m_{i1} - k\}, \dots, \max\{0, m_{it} - k\}) \mid 1 \leq i \leq r, k \in \mathbb{N}, m_{ij} = 0 \text{ if } (i, j) \notin D_{\mathbb{X}}\}$ and $\mathcal{S}_{\mathbb{V}}$ is defined in the same way. Again Corollary 4.10 in [8] yields that

$$\mathrm{HF}_{\mathbb{Y}}(i, j) = \sum_{k=0}^{\min\{i, l-1\}} c_{kj}.$$

Hence the conclusion follows from Corollary 3.7. \square

Remark 5.5. Let us observe that if the thickening \mathbb{V} of \mathbb{Y} is ACM then it follows that also \mathbb{Y} is ACM. The converse is not true in general, since if \mathbb{Y} is an ACM set of distinct points whose support is not a complete intersection then its thickening \mathbb{V} is not ACM. More precisely, if \mathbb{V} is a set of double points supported on an ACM set of distinct points which is not a complete intersection then it is known that it is not ACM (see [9]).



Proposition 5.6. *Let \mathbb{X} be a reduced CI(d_1, d_2) with $d_1 \leq d_2$.*

(i) *For $m \geq 1$, we have the following exact sequence of graded $R_{m\mathbb{X}}$ -modules*

$$0 \longrightarrow I_{\mathbb{X}}^m / I_{\mathbb{X}}^{m+1} \longrightarrow (S/I_{\mathbb{X}}^m)^2(-1, 0) \oplus (S/I_{\mathbb{X}}^m)^2(0, -1) \longrightarrow \Omega_{R_{m\mathbb{X}}/K}^1 \longrightarrow 0$$

[illegible][illegible]

and the first difference function of $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}$ is

$$\begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & -1 & -1 & 0 & \dots \\ 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & -1 & -1 & 0 & \dots \\ 2 & 4 & 4 & 4 & 4 & 4 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 4 & 4 & 4 & 4 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 4 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 4 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For equimultiple fat point schemes supported at a complete intersection, the Hilbert function of the Kähler differential module exhibits the following type of uniform behaviour.

Proposition 5.10. *Let \mathbb{X} be a $CI(d_1, d_2)$ consisting of $s = d_1 d_2$ distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $m \geq 2$. For $1 \leq i \leq d_1$ and $1 \leq j \leq d_2$, let \mathbb{Y}_{ij} be the fat point scheme $\mathbb{Y}_{ij} = \sum_{(k,l) \in D_{\mathbb{X}} \setminus \{(i,j)\}} m P_{kl} + (m-1) P_{ij}$ supported at \mathbb{X} . Then the Hilbert function of $\Omega_{R_{\mathbb{Y}_{ij}}/K}^1$ does not depend on the choice of (i, j) .*

Proof. Since the equimultiple fat point scheme $m\mathbb{X}$ is ACM, the fat point schemes \mathbb{Y}_{ij} are also ACM. In this case we have $D_{\mathbb{X}} = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$. So, Theorem 2.14 implies that

$$\deg_{m\mathbb{X}}(P_{ij}) = ((d_1 - 1, md_2 - 1), (2d_1 - 1, (m-1)d_2 - 1), \dots, (md_1 - 1, d_2 - 1))$$

for all $P_{ij} \in \mathbb{X}$. According to [12, Corollary 4.4 and Theorem 5.5], we have

$$\mathrm{HF}_{\mathbb{Y}_{ij}}(k, l) = \mathrm{HF}_{m\mathbb{X}}(k, l) - \#\{(k', l') \in \deg_{m\mathbb{X}}(P_{ij}) \mid (k', l') \preceq (k, l)\}$$

for all $(k, l) \in \mathbb{N}^2$, and thus the fat point schemes \mathbb{Y}_{ij} all have the same Hilbert function. Similarly, also the fat point schemes $\mathbb{V}_{ij} = \sum_{(k,l) \in D_{\mathbb{X}} \setminus \{(i,j)\}} (m+1) P_{kl} + m P_{ij}$ all have the same Hilbert function. By Theorem 3.5, the sequence of bigraded $R_{\mathbb{Y}_{ij}}$ -modules

$$0 \longrightarrow I_{\mathbb{Y}_{ij}}/I_{\mathbb{V}_{ij}} \longrightarrow R_{\mathbb{Y}_{ij}}^2(-1, 0) \oplus R_{\mathbb{Y}_{ij}}^2(0, -1) \longrightarrow \Omega_{R_{\mathbb{Y}_{ij}}/K}^1 \longrightarrow 0$$

is exact. Hence we have

$$\mathrm{HF}_{\Omega_{R_{\mathbb{Y}_{ij}}/K}^1}(k, l) = 2 \mathrm{HF}_{\mathbb{Y}_{ij}}(k-1, l) + 2 \mathrm{HF}_{\mathbb{Y}_{ij}}(k, l-1) + \mathrm{HF}_{\mathbb{Y}_{ij}}(k, l) - \mathrm{HF}_{\mathbb{V}_{ij}}(k, l)$$

for all (k, l) . The conclusion follows. \square

If we remove the hypothesis $m \geq 2$ in the preceding proposition, we are still able to show the following result for complete intersections \mathbb{X} of type (h, h) .

Proposition 5.11. *Let $h \geq 1$, let \mathbb{X} be a $CI(h, h)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and for $(i, j) \in D_{\mathbb{X}}$ let $\mathbb{X}_{ij} = \mathbb{X} \setminus \{P_{ij}\}$. Then the Hilbert function of $\Omega_{R_{\mathbb{X}_{ij}}/K}^1$ does not depend on the choice of (i, j) .*

Proof. For $i, j = 1, \dots, h$, we let \mathbb{W}_{ij} and \mathbb{Y}_{ij} be fat point schemes

$$\mathbb{W}_{ij} = \sum_{(k,l) \in D_{\mathbb{X}} \setminus \{(i,j)\}} 2P_{kl} + P_{ij} \quad \text{and} \quad \mathbb{Y}_{ij} = \sum_{(k,l) \in D_{\mathbb{X}} \setminus \{(i,j)\}} 2P_{kl}.$$

Note that $\text{Supp}(\mathbb{W}_{ij}) = \mathbb{X}$ and $\text{Supp}(\mathbb{Y}_{ij}) = \mathbb{X}_{ij}$. We can argue as in the proof of Proposition 5.10 to get that the Hilbert function of \mathbb{W}_{ij} does not depend on (i, j) . Moreover, since \mathbb{X} is complete intersection, Theorem 2.13 shows that all the subsets \mathbb{X}_{ij} have the same Hilbert function. Let $f^* \in R_{\mathbb{W}_{ij}}$ be the minimal separator of P_{ij} in \mathbb{W}_{ij} . By Theorem 2.13, it is unique since the multiplicity of P_{ij} in \mathbb{W}_{ij} is 1. Let $F^* \in S$ be a representative of f^* . Using Theorem 2.14, we have that $\deg(F^*) = (2(h-1), 2(h-1))$ and this bidegree does not depend on the choice of (i, j) . Since the support \mathbb{X} is a complete intersection, we can always renumber the lines such that we have $(i, j) = (h, h)$, i.e., $P_{ij} = P_{hh}$. Thus the Hilbert functions of the schemes \mathbb{Y}_{ij} do not depend of the choice of (i, j) . Therefore an application of Corollary 3.7 yields the desired property. \square

Example 5.12. Consider Example 5.9, and reduce the multiplicity of P_{11} by one, i.e., let $\mathbb{Y}_{11} = 2P_{11} + 3P_{12} + 3P_{13} + 3P_{21} + 3P_{22} + 3P_{23}$ be the subscheme of \mathbb{Y} . Then the first difference function $\Delta \text{HF}_{\Omega_{R_{\mathbb{Y}_{11}}/K}^1}$ is given by

$$\begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & -1 & -1 & 0 & \dots \\ 2 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & -1 & -1 & 0 & 0 & \dots \\ 2 & 4 & 4 & 4 & 4 & 4 & 3 & 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 4 & 4 & 4 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 4 & 3 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Moreover, every subscheme \mathbb{Y}_{ij} of \mathbb{Y} , obtained by reducing in $3\mathbb{X}$ the multiplicity of one point P_{ij} by one, satisfies that $\text{HF}_{\Omega_{R_{\mathbb{Y}_{ij}}/K}^1} = \text{HF}_{\Omega_{R_{\mathbb{Y}_{11}}/K}^1}$.

In the final part of this section we look at the Hilbert function of the Kähler differential module of an equimultiple fat point scheme whose support is an almost complete intersection.

Definition 5.13. A set of points \mathbb{X} in $\mathbb{P}^1 \times \mathbb{P}^1$ is called an *almost complete intersection* if the number of minimal generators of $I_{\mathbb{X}}$ is one more than the codimension, i.e., if $I_{\mathbb{X}}$ has exactly three minimal generators.

Remark 5.14. In [4], the authors give a complete description of a minimal bigraded free resolution of $R_{\mathbb{X}}$, where \mathbb{X} is an almost complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$.

Using [7, Theorem 1.1] and [2, Corollary 4.4] we have that

Theorem 5.15. Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be an ACM set of points. For all $m \geq 1$, we have $I_{\mathbb{X}}^m = I_{\mathbb{X}}^{(m)} \Leftrightarrow I_{\mathbb{X}}^3 = I_{\mathbb{X}}^{(3)} \Leftrightarrow I_{\mathbb{X}}$ is a complete intersection or $I_{\mathbb{X}}$ is an almost complete intersection.

Proposition 5.16. Let \mathbb{X} be an almost complete intersection of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose associated tuple is $\alpha_{\mathbb{X}} = (\underbrace{d_1, \dots, d_1}_a, \underbrace{d_2, \dots, d_2}_b)$ with $d_1 > d_2$.

- (i) For $m \geq 1$, we have the following exact sequence of bigraded $R_{m\mathbb{X}}$ -modules
- $$0 \longrightarrow I_{\mathbb{X}}^m / I_{\mathbb{X}}^{m+1} \longrightarrow (S/I_{\mathbb{X}}^m)^2(-1, 0) \oplus (S/I_{\mathbb{X}}^m)^2(0, -1) \longrightarrow \Omega_{R_{m\mathbb{X}}/K}^1 \longrightarrow 0$$

(ii) For all $j \in \mathbb{N}$, if $i \geq (m+1)(a+b) - 1$, we have

$$\mathrm{HF}_{\Omega_{R_{m\mathbb{X}}/K}^1}(i, j) = 4 \sum_{k=1}^j \alpha_k^* + 2\alpha_{j+1}^* - \sum_{k=1}^h \#\{a \in \hat{\alpha}_{m\mathbb{X}} \mid a \geq k\}$$

where $\alpha_{m\mathbb{X}}^* = (\alpha_1^*, \alpha_2^*, \dots)$ with $\alpha_i^* = \#\{\alpha_j \in \alpha_{m\mathbb{X}} \mid \alpha_j \geq i\}$, and where $h = \min\{j+1, (m+1)d_1\}$.

Proof. It follows from Theorem 5.15 that $I_{\mathbb{X}}^{(m)} = I_{\mathbb{X}}^m$. So, claim (i) follows from Theorem 3.5. Moreover, in this case the equimultiple fat point scheme $\mathbb{Y} = m\mathbb{X}$ is not ACM and its associated tuple is of the form

$$\alpha_{\mathbb{Y}} = (\underbrace{md_1, \dots, md_1}_a, \dots, \underbrace{d_2, \dots, d_2}_b).$$

In particular, we have $r = a + b$, $l = m(a + b)$, $t = d_1$ and $l' = md_1$. Hence we may apply Proposition 4.3 and get claim (ii). \square

Example 5.17. Let $i \geq 0$, let $Q_i = R_i = [1 : i] \in \mathbb{P}^1$, and let P_{ij} denote the point $Q_i \times R_j$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We let \mathbb{X} be the almost complete intersection $\mathbb{X} = \{P_{11}, P_{12}, P_{13}, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathbb{Y} = 3\mathbb{X}$ be the equimultiple fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ supported at \mathbb{X} . Then we have $\alpha_{\mathbb{X}} = (3, 3, 2)$ and $\alpha_{\mathbb{Y}} = (9, 9, 6, 6, 6, 6, 4, 3, 3, 2)$. Thus the Hilbert function $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}$ is given by

$$\begin{bmatrix} 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 19 & 18 & 18 & 18 \dots \\ 2 & 8 & 14 & 20 & 26 & 32 & 38 & 44 & 50 & 55 & 55 & 53 & 53 & 53 \dots \\ 4 & 14 & 24 & 34 & 44 & 54 & 64 & 74 & 84 & 89 & 86 & 84 & 84 & 84 \dots \\ 6 & 20 & 34 & 48 & 62 & 76 & 90 & 103 & 114 & 117 & 114 & 112 & 112 & 112 \dots \\ 8 & 26 & 44 & 62 & 80 & 98 & 115 & 127 & 136 & 137 & 135 & 133 & 133 & 133 \dots \\ 10 & 32 & 54 & 76 & 98 & 120 & 137 & 146 & 153 & 154 & 152 & 150 & 150 & 150 \dots \\ 12 & 38 & 64 & 90 & 115 & 138 & 153 & 161 & 166 & 168 & 166 & 164 & 164 & 164 \dots \\ 14 & 44 & 74 & 103 & 127 & 148 & 161 & 168 & 173 & 175 & 173 & 171 & 171 & 177 \dots \\ 16 & 50 & 84 & 113 & 134 & 153 & 166 & 173 & 179 & 181 & 179 & 177 & 177 & 177 \dots \\ 18 & 55 & 90 & 117 & 138 & 157 & 171 & 178 & 184 & 186 & 184 & 182 & 182 & 182 \dots \\ 19 & 55 & 89 & 116 & 138 & 157 & 171 & 178 & 184 & 186 & 184 & 182 & 182 & 182 \dots \\ 18 & 53 & 87 & 114 & 136 & 155 & 169 & 176 & 182 & 184 & 182 & 180 & 180 & 180 \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

6. THE KÄHLER DIFFERENT FOR A FAT POINT SCHEME IN $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we examine the initial Fitting ideal of the Kähler differential module, commonly called the Kähler different, for a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. In the following we restrict our attention to ACM fat point schemes \mathbb{Y} in $\mathbb{P}^1 \times \mathbb{P}^1$. In this case we may assume that x_0, y_0 is a regular sequence in $R_{\mathbb{Y}}$. We let $R_o = K[x_0, y_0]$. Then the algebra R/R_o is finite and the monomorphism $R_o \hookrightarrow R$ defines a Noether normalization.

Suppose that $\{F_1, \dots, F_r\}$ is a bihomogeneous set of generators of $I_{\mathbb{Y}}$. According to [20, Proposition 4.19], the Kähler differential module $\Omega_{R_{\mathbb{Y}}/R_o}^1$ has a presentation

$$0 \longrightarrow \mathcal{K} \longrightarrow R_{\mathbb{Y}} dX_1 \oplus R_{\mathbb{Y}} dY_1 \longrightarrow \Omega_{R_{\mathbb{Y}}/R_o}^1 \longrightarrow 0$$

where \mathcal{K} is generated by the elements $\frac{\partial F_i}{\partial x_1} dX_1 + \frac{\partial F_i}{\partial y_1} dY_1$ such that $j \in \{1, \dots, r\}$. The Jacobian matrix

$$\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_r}{\partial x_1} \\ \frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_r}{\partial y_1} \end{bmatrix}$$

is a relation matrix of $\Omega_{R_{\mathbb{Y}}/R_o}^1$ with respect to $\{dx_1, dy_1\}$. The initial Fitting ideal $F_0(\Omega_{R_{\mathbb{Y}}/R_o}^1)$ of $\Omega_{R_{\mathbb{Y}}/R_o}^1$ is the bihomogeneous ideal of $R_{\mathbb{Y}}$ generated by all 2-minors of the Jacobian matrix.

Definition 6.1. The initial Fitting ideal of $\Omega_{R_{\mathbb{Y}}/R_o}^1$ is denoted by

$$\vartheta_{\mathbb{Y}} = F_0(\Omega_{R_{\mathbb{Y}}/R_o}^1)$$

and is called the *Kähler different* of $R_{\mathbb{Y}}/R_o$ (or for \mathbb{Y} w.r.t. $\{x_0, y_0\}$).

Since the Kähler different is a bihomogeneous ideal in $R_{\mathbb{Y}}$, we can examine its bigraded Hilbert function. The following proposition provides some basic properties of this function.

Proposition 6.2. Let $\mathbb{Y} = \sum_{(i,j) \in D_{\mathbb{X}}} m_{ij} P_{ij}$ be an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ supported at \mathbb{X} .

- (i) For all $(i, j) \in \mathbb{N}^2$, we have $\text{HF}_{\vartheta_{\mathbb{Y}}}(i, j) \leq \text{HF}_{\vartheta_{\mathbb{Y}}}(i+1, j)$ and $\text{HF}_{\vartheta_{\mathbb{Y}}}(i, j) \leq \text{HF}_{\vartheta_{\mathbb{Y}}}(i, j+1)$.
- (ii) We have $\text{HF}_{\vartheta_{\mathbb{Y}}} = 0$ if and only if $m_{ij} \geq 2$ for all $P_{ij} \in \mathbb{X}$.
- (iii) Let s' be the number of points $P_{ij} \in \mathbb{X}$ such that $m_{ij} = 1$. Then we have $\text{HF}_{\vartheta_{\mathbb{Y}}}(i, j) = s'$ for all large enough $i, j \gg 0$.

Proof. (i) This follows from the fact that $\vartheta_{\mathbb{Y}}$ is a bihomogeneous ideal of $R_{\mathbb{Y}}$ and x_0, y_0 are non-zerodivisors of $R_{\mathbb{Y}}$.

(ii) Suppose that $m_{ij} \geq 2$ for all $(i, j) \in D_{\mathbb{X}}$. Write $\wp_{ij} = \langle L_{Q_i}, L_{R_j} \rangle$ with two linear forms $L_{Q_i} \in S_{1,0}$ and $L_{R_j} \in S_{0,1}$. Then $I_{\mathbb{Y}} = \bigcap_{(i,j) \in D_{\mathbb{X}}} \wp_{ij}^{m_{ij}}$. For any two bihomogeneous minimal generators F, G of $I_{\mathbb{Y}}$, it is easy to see that $\frac{\partial(F, G)}{\partial(X_1, Y_1)} \in \wp_{ij}^{m_{ij}}$ for all $(i, j) \in D_{\mathbb{X}}$. Since $\vartheta_{\mathbb{Y}}$ is generated by the images in $R_{\mathbb{Y}}$ of the elements of the form $\frac{\partial(F, G)}{\partial(X_1, Y_1)}$ where F, G are bihomogeneous minimal generators of $I_{\mathbb{Y}}$, we get $\vartheta_{\mathbb{Y}} = \langle 0 \rangle$.

Conversely, w.l.o.g. assume that $m_{i_0 j_0} = 1$. Let $F \in S$ be a separator for $P_{i_0 j_0}$. Obviously, we have $L_{Q_{i_0}} F, L_{R_{j_0}} F \in I_{\mathbb{Y}}$. We write $P_{i_0 j_0} = [a_0 : a_1] \times [b_0 : b_1]$ with $a_0 \neq 0$ and $b_0 \neq 0$, and so $L_{Q_{i_0}} = a_1 X_0 - a_0 X_1$ and $L_{R_{j_0}} = b_1 Y_0 - b_0 Y_1$. We also see that

$$\begin{aligned} \frac{\partial(L_{Q_{i_0}} F, L_{R_{j_0}} F)}{\partial(X_1, Y_1)} &= \det \begin{bmatrix} F \frac{\partial L_{Q_{i_0}}}{\partial X_1} + L_{Q_{i_0}} \frac{\partial F}{\partial X_1} & F \frac{\partial L_{R_{j_0}}}{\partial X_1} + L_{R_{j_0}} \frac{\partial F}{\partial X_1} \\ F \frac{\partial L_{Q_{i_0}}}{\partial Y_1} + L_{Q_{i_0}} \frac{\partial F}{\partial Y_1} & F \frac{\partial L_{R_{j_0}}}{\partial Y_1} + L_{R_{j_0}} \frac{\partial F}{\partial Y_1} \end{bmatrix} \\ &= F^2 \left(\frac{\partial L_{Q_{i_0}}}{\partial X_1} \frac{\partial L_{R_{j_0}}}{\partial Y_1} - \frac{\partial L_{Q_{i_0}}}{\partial Y_1} \frac{\partial L_{R_{j_0}}}{\partial X_1} \right) + G \\ &= a_0 b_0 F^2 + G \end{aligned}$$

for some $G \in I_{\mathbb{Y}}$. Since $a_0 b_0 \neq 0$, we obtain $\overline{F}^2 \in \vartheta_{\mathbb{Y}}$. Furthermore, F^2 is also a separator for $P_{i_0 j_0}$, and hence $\overline{F}^2 \neq 0$.

(iii) Let $I = \bigcap_{(i,j) \in D_{\mathbb{X}}, m_{ij} > 1} \wp_{ij}^{m_{ij}}$. As in the proof of (ii), we have $\vartheta_{\mathbb{Y}} \subseteq \wp_{ij}^{m_{ij}}/I_{\mathbb{Y}}$ if $m_{ij} > 1$, and so $\vartheta_{\mathbb{Y}} \subseteq I/I_{\mathbb{Y}}$. This implies

$$\text{HF}_{\vartheta_{\mathbb{Y}}}(i, j) \leq \text{HF}_{I/I_{\mathbb{Y}}}(i, j) = \text{HF}_{\mathbb{Y}}(i, j) - \text{HF}_{S/I}(i, j)$$

for all $(i, j) \in \mathbb{N}^2$. Let $r = \#\pi_1(\mathbb{X})$ and $t = \#\pi_2(\mathbb{X})$. Set $l = \sum_{i=1}^r \max\{m_{ie} \mid (i, e) \in D_{\mathbb{X}}\}$ and $l' = \sum_{j=1}^t \max\{m_{ej} \mid (e, j) \in D_{\mathbb{X}}\}$ (see also Notation 4.1). We have $\mathrm{HF}_{\mathbb{Y}}(i, j) = \sum_{(i, j) \in D_{\mathbb{X}}} \binom{m_{ij}+1}{2}$ and $\mathrm{HF}_{S/I}(i, j) = \sum_{(i, j) \in D_{\mathbb{X}}, m_{ij} > 1} \binom{m_{ij}+1}{2}$ for all $(i, j) \succeq (l, l')$, and so we get $\mathrm{HF}_{\vartheta_{\mathbb{Y}}}(i, j) \leq s'$ for $(i, j) \succeq (l, l')$. It follows from (i) that $\mathrm{HF}_{\vartheta_{\mathbb{Y}}}(i, j) \leq s'$ for all $(i, j) \in \mathbb{N}^2$. Moreover, for each point P_{ij} with $m_{ij} = 1$ and for which F_{ij} is a minimal separator for P_{ij} , we have $\overline{F}_{ij}^2 \in \vartheta_{\mathbb{Y}}$ by the same reasoning as in the proof of part (ii). According to Theorem 2.14, we know that $\deg(F_{ij}) = \left(\sum_{(e, j) \in D_{\mathbb{X}}} m_{ej} - 1, \sum_{(i, e) \in D_{\mathbb{X}}} m_{ie} - 1 \right)$. We set

$$t_1 = 2 \max \left\{ \sum_{(e, j) \in D_{\mathbb{X}}} m_{ej} - 1 \mid (i, j) \in D_{\mathbb{X}}, m_{ij} = 1 \right\}$$

and

$$t_2 = 2 \max \left\{ \sum_{(i, e) \in D_{\mathbb{X}}} m_{ie} - 1 \mid (i, j) \in D_{\mathbb{X}}, m_{ij} = 1 \right\}.$$

Since F_{ij}^2 is also a separator for P_{ij} , we have

$$s' = \dim_K(\langle \overline{F}_{ij}^2 \mid (i, j) \in D_{\mathbb{X}}, m_{ij} = 1 \rangle)_{i', j'} \leq \mathrm{HF}_{\vartheta_{\mathbb{Y}}}(i', j') \leq s'$$

for all $(i', j') \succeq (t_1, t_2)$. Hence the equality $\mathrm{HF}_{\vartheta_{\mathbb{Y}}}(i, j) = s'$ holds true for all $(i, j) \succeq (t_1, t_2)$. \square

If all multiplicities are 1, i.e., in the case of an ACM set of reduced points, we can improve the description of the Hilbert function of $\vartheta_{\mathbb{X}}$ as follows.

Proposition 6.3. *Let \mathbb{X} be an ACM set of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$, let $r = \#\pi_1(\mathbb{X})$, and let $t = \#\pi_2(\mathbb{X})$.*

- (i) *If $r \geq 2$, then $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(0, j) = 0$ for $j < t - 1$.*
- (ii) *If $t \geq 2$, then $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i, 0) = 0$ for $i < r - 1$.*
- (iii) *There exists a non-zero divisor of $R_{\mathbb{X}}$ which is contained in $\vartheta_{\mathbb{X}}$.*
- (iv) *Let $j \in \mathbb{N}$, let $\{h_1, \dots, h_u\}$ be a bihomogeneous minimal system of generators of $\vartheta_{\mathbb{X}}$, let $\deg(h_k) = (i_k, j_k)$ for $k = 1, \dots, u$, and let $i_0 = \max\{i_k \mid j_k \leq j, k = 1, \dots, u\}$. For $i \geq i_0$, if $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i, j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i + 1, j)$ then $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i + 1, j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i + 2, j)$.*
- (v) *We have $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i, j) = s$ for all $(i, j) \succeq (2r - 2, 2t - 2)$.*

Proof. First we prove claim (i). Then claim (ii) follows similarly. Note that $\pi_2(\mathbb{X}) = \{R_1, \dots, R_t\} \subseteq \mathbb{P}^1$ is a complete intersection and $I_{\pi_2(\mathbb{X})} = \langle F \rangle \subseteq K[Y_0, Y_1]$ where $F = L_{R_1} \cdots L_{R_t}$ and L_{R_j} is the $(0, 1)$ -form that vanishes at R_j . Let $\alpha_{\mathbb{X}} = (\alpha_1, \dots, \alpha_r)$ be the associated tuple of \mathbb{X} . Since \mathbb{X} is ACM, we have $\alpha_1 = t$ or $\mathbb{X}_{Q_1} := \pi_1^{-1}(Q_1) = \{Q_1 \times R_1, \dots, Q_1 \times R_t\}$. Then $I_{\mathbb{X}_{Q_1}} = \langle L_{Q_1}, F \rangle$. Set $\mathbb{V} = \mathbb{X} \setminus \mathbb{X}_{Q_1}$. Clearly, $F \in I_{\mathbb{X}} = I_{\mathbb{V}} \cap I_{\mathbb{X}_{Q_1}}$.

We want to show that $I_{\mathbb{X}} = L_{Q_1} I_{\mathbb{V}} + \langle F \rangle$. It is clear that $L_{Q_1} I_{\mathbb{V}} + \langle F \rangle \subseteq I_{\mathbb{X}}$. Let $G \in I_{\mathbb{X}}$. Then $G = L_{Q_1} H_1 + F H_2$ with $H_1, H_2 \in S$. Since $G, F \in I_{\mathbb{X}}$, we have $L_{Q_1} H_1 \in I_{\mathbb{X}} \subseteq I_{\mathbb{V}}$, and so $(L_{Q_1} H_1)(Q \times R) = 0$ for all $Q \times R \in \mathbb{V}$. But $Q \neq Q_1$ for every $Q \times R \in \mathbb{V}$, this implies $H_1(Q \times R) = 0$ for every $Q \times R \in \mathbb{V}$. Thus $H_1 \in I_{\mathbb{V}}$.

If $r = 2$, then $I_{\mathbb{V}} = I_{\mathbb{X}_{Q_2}} = \langle L_{Q_2}, G \rangle$ for some $G \in K[Y_0, Y_1]$. Then $I_{\mathbb{X}} = \langle L_{Q_1} L_{Q_2}, L_{Q_1} G, F \rangle$. The element $\frac{\partial(L_{Q_1} G, F)}{\partial(x_1, y_1)}$ has degree $(0, t - 1) + \deg(G) = (0, j_0)$ and any other bihomogeneous element of $\vartheta_{\mathbb{X}}$ has degree $(0, j)$ with $j \geq j_0 > t - 1$.

Hence we get $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(0, j) = 0$ for $j \leq t - 1$. In the case $r > 2$ we also have the above equality, since $I_V \subseteq I_{\mathbb{X}_{Q_2}} = \langle L_{Q_2}, G \rangle$.

Now we prove (iii). Since $\pi_1(\mathbb{X})$ and $\pi_2(\mathbb{X})$ are complete intersections in \mathbb{P}^1 , we write $I_{\pi_1(\mathbb{X})} = \langle F_1 \rangle \subseteq K[X_0, X_1]$ and $I_{\pi_2(\mathbb{X})} = \langle F_2 \rangle \subseteq K[Y_0, Y_1]$, where F_1 and F_2 are bihomogeneous. Then $\frac{\partial(F_1, F_2)}{\partial(x_1, y_1)} = \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial y_1} \in \vartheta_{\mathbb{X}}$. Note that $\frac{\partial F_1}{\partial x_1}(Q_i) \neq 0$ for all $Q_i \in \pi_1(\mathbb{X})$ and $\frac{\partial F_2}{\partial y_1}(R_j) \neq 0$ for all $R_j \in \pi_2(\mathbb{X})$. Thus the element $\frac{\partial(F_1, F_2)}{\partial(x_1, y_1)}$ is a non-zero-divisor of $R_{\mathbb{X}}$.

For (iv), suppose that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i, j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1, j)$. Because x_0 is a non-zero-divisor of $R_{\mathbb{X}}$, the multiplication map $(\vartheta_{\mathbb{X}})_{i,j} \xrightarrow{\mu_{x_0}} (\vartheta_{\mathbb{X}})_{i+1,j}$ is an isomorphism of K -vector spaces. So, we have $(\vartheta_{\mathbb{X}})_{i+1,j} = x_0 \cdot (\vartheta_{\mathbb{X}})_{i,j}$. Obviously, $x_0 \cdot (\vartheta_{\mathbb{X}})_{i+1,j} \subseteq (\vartheta_{\mathbb{X}})_{i+2,j}$. For the other inclusion, let $f \in (\vartheta_{\mathbb{X}})_{i+2,j} \setminus \{0\}$. Since $i \geq i_0$, we may write $f = x_0 f_0 + x_1 f_1$ where $f_0, f_1 \in (\vartheta_{\mathbb{X}})_{i+1,j}$. We write $f_k = x_0 g_k \in x_0 \cdot (\vartheta_{\mathbb{X}})_{i,j}$ for $k = 0, 1$. This implies

$$f = x_0 f_0 + x_1 f_1 = x_0(x_0 g_0 + x_1 g_1) \in x_0 \cdot (\vartheta_{\mathbb{X}})_{i+1,j}.$$

Hence $x_0 \cdot (\vartheta_{\mathbb{X}})_{i+1,j} = (\vartheta_{\mathbb{X}})_{i+2,j}$, and consequently $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+1, j) = \mathrm{HF}_{\vartheta_{\mathbb{X}}}(i+2, j)$.

Finally, claim (v) follows from the preceding proposition. \square

Example 6.4. Let $i \geq 0$, let $Q_i = R_i = [1 : i] \in \mathbb{P}^1$, and let P_{ij} denote the point $Q_i \times R_j$ in $\mathbb{P}^1 \times \mathbb{P}^1$. We let \mathbb{X} be the set of points $\mathbb{X} = \{P_{11}, P_{12}, P_{13}, P_{21}, P_{22}\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Then \mathbb{X} is ACM and $r = 2$ and $t = 3$. Also, the Hilbert function of $\vartheta_{\mathbb{X}}$ is

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 2 & 3 & 3 & \dots \\ 0 & 1 & 3 & 4 & 5 & 5 & \dots \\ 0 & 1 & 3 & 4 & 5 & 5 & \dots \\ 0 & 1 & 3 & 4 & 5 & 5 & \dots \\ 0 & 1 & 3 & 4 & 5 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

or equivalently, the first difference function is

$$\Delta \mathrm{HF}_{\vartheta_{\mathbb{X}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In this case we see that $\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i, j) = 5$ for all $(i, j) \succeq (2r - 2, 2t - 2) = (2, 4)$.

7. THE CAYLEY-BACHARACH PROPERTY

In this section we consider sets of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$ having the Cayley-Bacharach property.

First we recall this property for a set of reduced points $\mathbb{X} = \{P_1, \dots, P_s\}$ in \mathbb{P}^n . Let $r_{\mathbb{X}}$ be the regularity index of \mathbb{X} , i.e., the least degree such that the Hilbert function of \mathbb{X} equals the degree of \mathbb{X} . The set \mathbb{X} is called a *Cayley-Bacharach scheme* if every hypersurface of degree $r_{\mathbb{X}} - 1$ which contains all but one point of \mathbb{X} must contain all points of \mathbb{X} .

Notice that a set of distinct points $\mathbb{X} \subseteq \mathbb{P}^n$ is a Cayley-Bacharach scheme if and only if $\mathrm{HF}_{\mathbb{X} \setminus P_j}$ does not depend on the choice of j . Moreover, one can detect a Cayley-Bacharach scheme (especially, a Cayley-Bacharach scheme being a complete intersection) by looking at a particular homogeneous component of its Kähler different (see [16, Lemma 3.7] and [17, Theorem 5.6]).

In Section 5, Propositions 5.10 and 5.11 we showed that the Hilbert function of the Kähler differential module for subschemes obtained by reducing by one the multiplicity of one point P_{ij} in an equimultiple fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ supported on a complete intersection does not depend on the choice of (i, j) . For reduced schemes, this leads us to examine the Cayley-Bacharach property in $\mathbb{P}^1 \times \mathbb{P}^1$ which is defined as follows (see also [6]).

Definition 7.1. Let \mathbb{X} be a set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$. We say that \mathbb{X} has the *Cayley-Bacharach property* if the Hilbert function of $\mathbb{X} \setminus \{P_{ij}\}$ is independent of the choice of $P_{ij} \in \mathbb{X}$.

Remark 7.2. For any finite set $\Sigma \subseteq \mathbb{N}^2$ we set

$$D_\Sigma = \bigcup_{(i,j) \in \Sigma} \{ (k, l) \in \mathbb{N}^2 \mid (k, l) \succeq (i, j) \}.$$

If \mathbb{X} is a set of distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $P_{ij} \in \mathbb{X}$, [11, Theorem 2.2] shows that

$$\mathrm{HF}_{\mathbb{X} \setminus \{P_{ij}\}}(i, j) = \begin{cases} \mathrm{HF}_{\mathbb{X}}(i, j) & \text{if } (i, j) \notin D_{\deg_{\mathbb{X}}(P_{ij})}, \\ \mathrm{HF}_{\mathbb{X}}(i, j) - 1 & \text{if } (i, j) \in D_{\deg_{\mathbb{X}}(P_{ij})}. \end{cases}$$

Thus we can say that \mathbb{X} has the Cayley-Bacharach property if and only if all of its points have the same minimal separator degree.

From Theorem 2.14 we see that an ACM set of points $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with the associated tuples $\alpha_{\mathbb{X}} = (\alpha_1, \dots, \alpha_r)$ and $\beta_{\mathbb{X}} = (\beta_1, \dots, \beta_t)$ satisfies

$$\deg_{\mathbb{X}}(P_{ij}) = (\beta_j - 1, \alpha_i - 1)$$

for all $P_{ij} \in \mathbb{X}$. So, Remark 7.2 yields that \mathbb{X} has the Cayley-Bacharach property if and only if $\alpha_1 = \dots = \alpha_r = t$ and $\beta_1 = \dots = \beta_t = r$. By Theorem 2.13 (or [11, Theorem 4.1]), this is also equivalent to the fact that \mathbb{X} is a complete intersection.

Proposition 7.3. Let \mathbb{X} be an ACM set of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $r = \#\pi_1(\mathbb{X})$ and $t = \#\pi_2(\mathbb{X})$. The following statements are equivalent:

- (i) \mathbb{X} has the Cayley-Bacharach property.
- (ii) \mathbb{X} is a complete intersection.
- (iii) For every point $P_{ij} \in \mathbb{X}$, the Kähler different $\vartheta_{\mathbb{X}}$ contains no separator for P_{ij} of degree $\prec (2r - 2, 2t - 2)$.

If one of these conditions is satisfied, the Hilbert function of $\vartheta_{\mathbb{X}}$ is given by

$$\mathrm{HF}_{\vartheta_{\mathbb{X}}}(i, j) = \mathrm{HF}_{\mathbb{X}}(i - r + 1, j - t + 1)$$

for all $(i, j) \in \mathbb{N}^2$.

Proof. The equivalence of (i) and (ii) follows from the above argument. So, it suffices to show that these claims are equivalent to (iii). Suppose that \mathbb{X} is a $CI(d_1, d_2)$. We have $d_1 = r$, $d_2 = t$ and $I_{\mathbb{X}} = \langle F_1, F_2 \rangle$ with $F_1 \in S_{d_1, 0}$ and $F_2 \in S_{0, d_2}$. Then we get $\vartheta_{\mathbb{X}} = \langle \frac{\partial(F_1, F_2)}{\partial(x_1, y_1)} \rangle$. As in the proof of (iii) of Proposition 6.3, the element $\frac{\partial(F_1, F_2)}{\partial(x_1, y_1)}$ is a non-zerodivisor of $R_{\mathbb{X}}$. Moreover, every point $P_{ij} \in \mathbb{X}$ has

degree $\deg_{\mathbb{X}}(P_{ij}) = (r-1, t-1)$. If $\vartheta_{\mathbb{X}}$ contains a separator \overline{F}_{ij} for some P_{ij} of degree $\prec (2r-2, 2t-2)$. Then $F_{ij} = \frac{\partial(F_1, F_2)}{\partial(X_1, X_1)} \cdot H_{ij}$ with $\deg(H_{ij}) \prec (r-1, t-1)$. But in this case H_{ij} is also a separator for P_{ij} , and so $\deg_{\mathbb{X}}(P_{ij}) \prec (r-1, t-1)$, a contradiction.

Conversely, suppose that $\vartheta_{\mathbb{X}}$ contains no separator for P_{ij} of degree $\prec (2r-2, 2t-2)$ for all $P_{ij} \in \mathbb{X}$ and that \mathbb{X} does not have the Cayley-Bacharach property. Then there is a point $P_{ij} \in \mathbb{X}$ such that $\deg_{\mathbb{X}}(P_{ij}) \prec (r-1, t-1)$. Let F_{ij} be a minimal separator for P_{ij} of degree $(i, j) \prec (r-1, t-1)$. As in the proof of Proposition 6.2, we have $\overline{F}_{ij}^2 \in \vartheta_{\mathbb{X}}$ and $\deg(\overline{F}_{ij}^2) = (2i, 2j) \prec (2r-2, 2t-2)$. Furthermore, \overline{F}_{ij}^2 is also a separator for P_{ij} . This is a contradiction. \square

As an immediate consequence of the previous proposition and [8, Theorem 4.2], the first difference function of the Hilbert function of the Kähler different of a complete intersection \mathbb{X} in $\mathbb{P}^1 \times \mathbb{P}^1$ can be described explicitly as in our next corollary.

Corollary 7.4. *Suppose \mathbb{X} is a CI(d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^1$ with $d_1 \leq d_2$. Then we have*

$$\Delta \text{HF}_{\vartheta_{\mathbb{X}}} = \begin{bmatrix} d_1-1 \left\{ \begin{array}{cccccccccccccccc} 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \dots \\ \vdots & \ddots & \ddots & & & \ddots & \ddots & & & \ddots & \ddots & & \\ 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 0 & \dots & \dots & 0 & \dots \\ \vdots & \ddots & \ddots & & & \ddots & \ddots & & & \ddots & \ddots & & \\ 0 & \dots & \dots & 0 & 1 & 1 & \dots & 1 & 0 & \dots & \dots & 0 & \dots \end{array} \right. \\ \underbrace{\hspace{10em}}_{d_2-1} \quad \underbrace{\hspace{10em}}_{d_2} \\ \begin{array}{cccccccccccccccc} 0 & \dots & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & \dots \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \ddots \end{array} \end{bmatrix}.$$

Example 7.5. Let \mathbb{X} be a complete intersection of type $(4, 6)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The first difference function of $\text{HF}_{\vartheta_{\mathbb{X}}}$ is

$$\Delta \text{HF}_{\vartheta_{\mathbb{X}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.$$

REFERENCES

- [1] The ApCoCoA Team, *ApCoCoA: Approximate Computations in Commutative Algebra*, available at <http://www.apcocoa.org>.
- [2] S. Cooper, G. Fatabbi, E. Guardo, B. Harbourne, A. Lorenzini, J. Migliore, U. Nagel, A. Seceleanu, J. Szpond, A. Van Tuyl, Symbolic powers of codimension two Cohen-Macaulay ideals, Preprint 2016, available at <http://arxiv.org/pdf/1606.00935.pdf>.
- [3] G. de Dominicis, M. Kreuzer, Kähler differentials for points in \mathbb{P}^n , *J. Pure Appl. Algebra* **141** (1999), 153–173.

- [4] G. Favacchio, E. Guardo, The minimal free resolution of fat almost complete intersections in $\mathbb{P}^1 \times \mathbb{P}^1$, *Canad. J. Math.* <http://dx.doi.org/10.4153/CJM-2016-040-4>, Available on line November 2 (2016).
- [5] S. Giuffrida, R. Maggioni, A. Ragusa, On the postulation of 0-dimensional subschemes on a smooth quadric, *Pacific J. Math.* **155** (1992), 251–282.
- [6] E. Guardo, Schemi di punti grassi, Ph.D. Thesis, Università di Messina, 2000.
- [7] E. Guardo, B. Harbourne, A. Van Tuyl, Fat lines in \mathbb{P}^3 : powers versus symbolic powers, *J. Algebra* **390** (2013), 221–230.
- [8] E. Guardo, A. Van Tuyl, Fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their Hilbert functions, *Canad. J. Math.* **56** (2004), 716–741.
- [9] E. Guardo, A. Van Tuyl, The minimal resolutions of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support, *J. Pure Appl. Algebra* **211** (2007), 784–800.
- [10] E. Guardo, A. Van Tuyl, ACM sets of points in multiprojective space, *Collect. Math.* **59** (2008), 191–213.
- [11] E. Guardo, A. Van Tuyl, Separators of points in a multiprojective space, *Manuscripta Math.* **126** (2008), 99–113.
- [12] E. Guardo, A. Van Tuyl, Separators of fat points in $\mathbb{P}^n \times \mathbb{P}^m$, *J. Pure Appl. Algebra* **215** (2011), 1990–1998.
- [13] E. Guardo, A. Van Tuyl, Classifying ACM sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ via separators, *Arch. Math. (Basel)* **99** (2012), 33–36.
- [14] E. Guardo, A. Van Tuyl, Separators of arithmetically Cohen-Macaulay fat points in $\mathbb{P}^1 \times \mathbb{P}^1$, *J. Commut. Algebra* **4** (2012), 255–268.
- [15] E. Guardo, A. Van Tuyl, Arithmetically Cohen-Macaulay sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, *Springer Briefs in Mathematics*, Springer, Heidelberg, 2015.
- [16] M. Kreuzer, T.N.K. Linh, L.N. Long, Kähler differentials and Kähler differents for fat point schemes, *J. Pure Appl. Algebra* **219** (2015), 4479–4509.
- [17] M. Kreuzer, L.N. Long, Characterizations of zero-dimensional complete intersections, *Beiträge Algebra Geom.* DOI:10.1007/s13366-016-0311-9, Available on line July 29 (2016).
- [18] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 1*, Springer-Verlag, Heidelberg, 2000.
- [19] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 2*, Springer-Verlag, Heidelberg, 2005.
- [20] E. Kunz, *Kähler Differentials*, *Adv. Lectures Math.*, Vieweg Verlag, Braunschweig, 1986.
- [21] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.
- [22] O. Zariski, P. Samuel, *Commutative Algebra*, vol. II, *The University Series in Higher Mathematics*, D. Van Nostrand Co., Inc., Princeton, 1960.

DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIALE A. DORIA, 6, 95100 - CATANIA, ITALY
E-mail address: `guardo@dmf.unict.it`

FAKULTÄT FÜR INFORMATIK UND MATHEMATIK, UNIVERSITÄT PASSAU, D-94030 PASSAU, GERMANY
E-mail address: `martin.kreuzer@uni-passau.de`

DEPARTMENT OF MATHEMATICS, HUE UNIVERSITY'S COLLEGE OF EDUCATION, 34 LE LOI, HUE, VIETNAM
E-mail address: `tnkhanhlinh141@gmail.com`

FAKULTÄT FÜR INFORMATIK UND MATHEMATIK, UNIVERSITÄT PASSAU, D-94030 PASSAU, GERMANY, AND DEPARTMENT OF MATHEMATICS, HUE UNIVERSITY'S COLLEGE OF EDUCATION, 34 LE LOI, HUE, VIETNAM
E-mail address: `nghlong16633@gmail.com`